Meditationes Analyticae.

By
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OF BALLIOL COLLEGE, OXFORD.

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TO THOSE

WHO

LOVE TO WANDER ON THE SHORE

TILL THE DAY

WHEN THEIR EYES SHALL BE OPENED

AND THEY SHALL SEE CLEARLY

THE WORKS OF GOD

IN THE UNFATHOMED OCEAN OF TRUTH,

These Papers

ARE INSCRIBED.

PREFACE.

The following Papers have been written at various periods, as the subjects presented themselves to notice from time to time. If leisure had been afforded, an attempt would have been made to draw some of them up into a distinct treatise; but it was thought that even in their present form they might interest some of those who take pleasure in the pursuit of Mathematical Science. Some of the papers are entirely original; some are partly taken from foreign *Memoires*, and these chiefly from the writings of M. Cauchy.

In the solution of the problems presented to notice analysis has been almost invariably employed; the comprehensiveness and uniformity of that method being sufficient apology for the exclusion of geometrical proofs. In the form of the equations symmetry has been preserved, wherever the circumstances of the case would permit; for although such expressions are sometimes longer than the corresponding unsymmetrical ones, yet being more readily committed to memory, more expressive of their meaning, and to one familiar with them more easily applicable, they have been thought worthy of attention.

The following theorems, on account of their frequent occurrence in the following pages, are transcribed from Gregory's Solid Geometry.

I. If
$$\frac{a}{b} = \frac{a_1}{b_1} = \frac{a_2}{b_2} = \&c.$$
each of these ratios is equal to
$$\frac{(a^2 + a_1^2 + a_2^2 + \&c.)^{\frac{1}{2}}}{(b^2 + b_1^2 + b_2^2 + \&c.)^{\frac{1}{2}}},$$
and to
$$\frac{na + n_1a_1 + n_2a_2 + \&c.}{nb + n_1b_1 + n_2b_2 + \&c.};$$
n, n_1, n_2, &c. being any quantities whatever.

For assuming each of the ratios equal to r, we have
$$a = rb, \quad a_1 = rb_1, \quad a_2 = rb_2, \ \&c.$$
Squaring and adding,
$$a^2 + a_1^2 + a_2^2 + \&c. = r^2(b^2 + b_1^2 + b_2^2 + \&c.):$$
whence, extracting the root and dividing,
$$\frac{(a^2 + a_1^2 + a_2^2 + \&c.)^{\frac{1}{2}}}{(b^2 + b_1^2 + b_2^2 + \&c.)^{\frac{1}{2}}} = r = \frac{a}{b} = \frac{a_1}{b_1} = \&c.$$

Again, $\quad na = rnb, \quad n_1a_1 = rn_1b_1, \quad n_2a_2 = rn_2b_2, \ \&c.$

By addition,
$$(na + n_1a_1 + n_2a_2 + \&c.) = r(nb + n_1b_1 + n_2b_2 + \&c.)$$
Whence
$$\frac{na + n_1a_1 + n_2a_2 + \&c.}{nb + n_1b_1 + n_2b_2 + \&c.} = r = \frac{a}{b} = \frac{a_1}{b_1} = \&c.$$

II. If we wish to determine the variables from three simultaneous equations of the form
$$ax + by + cz = d \ldots\ldots (1),$$
$$a_1x + b_1y + c_1z = d_1 \ldots\ldots (2),$$
$$a_2x + b_2y + c_2z = d_2 \ldots\ldots (3),$$
instead of eliminating first z and then y, in order to determine x, we may eliminate both at one operation by the following rule: Multiply

(1) by $b_1c_2 - c_1b_2$; (2) by $cb_2 - bc_2$; (3) by $bc_1 - b_1c$, and add: it will be found that the coefficients of y and z are identically equal to zero, and we have

$$x = \frac{d(b_1c_2 - c_1b_2) + d_1(cb_2 - bc_2) + d_2(bc_1 - b_1c)}{a(b_1c_2 - c_1b_2) + a_1(cb_2 - bc_2) + a_2(bc_1 - b_1c)};$$

with similar expressions for the other variables. If $d=0$, $d_1=0$, $d_2=0$, the equations contain only two independent variables, since we may divide all by any one of them; and the condition that the equations should coexist is

$$a(b_1c_2 - c_1b_2) + a_1(cb_2 - bc_2) + a_2(bc_1 - b_1c) = 0.$$

We shall frequently refer to this process under the name of *cross multiplication;* and the student is recommended to make himself familiar with the forms of the multipliers, as a ready use of the process will be of great service to him.

III. The sum of three squares of the form

$$(ay - bx)^2 + (cx - az)^2 + (bz - cy)^2$$

may be put in a shape which is very convenient, especially in geometrical investigations. For if we add and substract from the preceding expression the three squares

$$a^2x^2, \quad b^2y^2, \quad c^2z^2,$$

the expression may be transformed into

$$(a^2 + b^2 + c^2)\ (x^2 + y^2 + z^2) - (ax + by + cz)^2,$$

or $$(a^2 + b^2 + c^2)\ (x^2 + y^2 + z^2)\left\{1 - \frac{(ax + by + cz)^2}{(a^2 + b^2 + c^2)\ (x^2 + y^2 + z^2)}\right\}.$$

Now, if a, b, c be taken as proportional to the direction-cosines of some one line, and x, y, z of another, the expression

$$\frac{ax + by + cz}{(a^2 + b^2 + c^2)^{\frac{1}{2}}(x^2 + y^2 + z^2)^{\frac{1}{2}}}$$

is equal to the cosine of the angle between the lines: let this be θ; then the sum of the squares is equal to

$$(a^2 + b^2 + c^2)\ (x^2 + y^2 + z^2)\ (\sin\theta)^2.$$

IV. If we obtain as the result of any process that a function of x is equal to a function of y in which y is involved in a manner similar to that in which x is involved in the other, then, as there is nothing to distinguish one co-ordinate from another when they are symmetrically involved, we may say that each of these functions is equal to a similar function of z, and this is the consequence of the general symmetry of our expressions. Thus, if we have two equations

$$lx + my + nz = 0,$$
$$l'x + m'y + n'z = 0,$$

and eliminate z between them, we find

$$(ln' - l'n)\,x + (mn' - m'n)\,y = 0,$$

or
$$\frac{x}{mn' - m'n} = \frac{y}{l'n - ln'};$$

here the two sides of the equation are symmetrical, one with respect to x and the other to y. We may therefore say that each is equal to $\frac{z}{lm' - l'm}$, this being the corresponding symmetrical expression with respect to z.

Balliol College, Jan. 1, 1847.

MEDITATIONES ANALYTICÆ.

Symmetrical Investigations of Formulæ relative to Plane Triangles.

In any plane triangle we have the relations

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{1}$$

$$= \frac{\sin C \cos B + \sin B \cos C}{c \cos B + b \cos C} = \frac{\sin A \cos C + \sin C \cos A}{a \cos C + c \cos A} = \frac{\sin B \cos A + \sin A \cos B}{b \cos A + a \cos B} \tag{2}$$

$$= \frac{\sin (B + C)}{c \cos B + b \cos C} = \frac{\sin (C + A)}{a \cos C + c \cos A} = \frac{\sin (A + B)}{b \cos A + a \cos B} \tag{3}$$

$$= \frac{\sin A}{c \cos B + b \cos C} = \frac{\sin B}{a \cos C + c \cos A} = \frac{\sin C}{b \cos A + a \cos B} \tag{4}$$

whence

$$\frac{c}{a} \cos B + \frac{b}{a} \cos C = \frac{a}{b} \cos C + \frac{c}{b} \cos A = \frac{b}{a} \cos A + \frac{a}{c} \cos B \tag{5}$$

whence

$$\left.\begin{aligned} \frac{\left(\frac{c}{b} - \frac{b}{c}\right)}{a} \cos A &= \frac{\cos B}{c} - \frac{\cos C}{b} \\ \frac{\left(\frac{a}{c} - \frac{c}{a}\right)}{b} \cos B &= \frac{\cos C}{a} - \frac{\cos A}{c} \\ \frac{\left(\frac{b}{a} - \frac{a}{b}\right)}{c} \cos C &= \frac{\cos A}{b} - \frac{\cos B}{a} \end{aligned}\right\} \tag{6}$$

whence, multiplying by

$$bc, \quad ca, \quad ab,$$

respectively, there result

$$\left.\begin{aligned} \frac{b^2 - c^2}{a} \cos A &= b \cos B - c \cos C \\ \frac{c^2 - a^2}{b} \cos B &= c \cos C - a \cos A \\ \frac{a^2 - b^2}{c} \cos C &= a \cos A - b \cos B \end{aligned}\right\} \tag{7}$$

whence

$$(b^2 - c^2) \cos^2 A + (c^2 - a^2) \cos^2 B + (a^2 - b^2) \cos^2 C = 0 \tag{8}$$

and

$$\frac{b^2 - c^2}{a} \cos A + \frac{c^2 - a^2}{b} \cos B + \frac{a^2 - b^2}{c} \cos C = 0 \tag{9}$$

and

$$\frac{\cos A}{a(b^2 + c^2 - a^2)} = \frac{\cos B}{b(c^2 + a^2 - b^2)} = \frac{\cos C}{c(a^2 + b^2 - c^2)} \tag{10}$$

also, since

$$\left.\begin{aligned} \frac{\sin A \cos A}{a \cos A} &= \frac{\sin B \cos B}{b \cos B} = \frac{\sin C \cos C}{c \cos C} \\ &= \frac{\sin 2A}{2a \cos A} = \frac{\sin 2B}{2b \cos B} = \frac{\sin 2C}{2c \cos C} \end{aligned}\right\} \tag{11}$$

consequently

$$\frac{b^2 - c^2}{a^2} \sin 2A + \frac{c^2 - a^2}{b^2} \sin 2B + \frac{a^2 - b^2}{c^2} \sin 2C = 0 \tag{12}$$

$$\frac{\sin 2A}{a^2(b^2 + c^2 - a^2)} = \frac{\sin 2B}{b^2(c^2 + a^2 - b^2)} = \frac{\sin 2C}{c^2(a^2 + b^2 - c^2)} \tag{13}$$

also by geometry each of the ratios (10) is

$$= \frac{1}{2abc} \tag{14}$$

hence

$$\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc} \tag{15}$$

$$a \cos A + b \cos B + c \cos C = \frac{2(b^2c^2 + c^2a^2 + a^2b^2) - (a^4 + b^4 + c^4)}{2abc} \tag{16}$$

$$- \left(\frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c}\right) - \frac{1}{2}\left(\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{a^3}{ab}\right)$$

and similarly other formulæ might be obtained.

Again, writing

$$b^2 + c^2 - a^2 = \lambda^2 \qquad c^2 + a^2 - b^2 = \mu^2 \qquad a^2 + b^2 - c^2 = \nu^2 \tag{17}$$

we find by means of (1) and (10)

$$\lambda^2 \tan A = \mu^2 \tan B = \nu^2 \tan C \tag{18}$$

also, since in every plane triangle

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C \tag{19}$$

$$\therefore \frac{\tan A}{\frac{1}{\lambda^2}} = \frac{\tan B}{\frac{1}{\mu^2}} = \frac{\tan C}{\frac{1}{\nu^2}} = \frac{\tan A + \tan B + \tan C}{\frac{1}{\lambda^2} + \frac{1}{\mu^2} + \frac{1}{\nu^2}} \tag{20}$$

$$= \frac{\tan A \tan B \tan C}{\frac{1}{\lambda^2} + \frac{1}{\mu^2} + \frac{1}{\nu^2}} = \frac{\lambda^2 \tan A\, \mu^2 \tan B\, \nu^2 \tan C}{\mu^2\nu^2 + \nu^2\lambda^2 + \lambda^2\mu^2} \tag{21}$$

$$= \frac{\lambda^6 \tan^3 A}{\mu^2\nu^2 + \nu^2\lambda^2 + \lambda^2\mu^2} = \frac{\mu^6 \tan^3 B}{\mu^2\nu^2 + \nu^2\lambda^2 + \lambda^2\mu^2} = \frac{\nu^6 \tan^3 C}{\mu^2\nu^2 + \nu^2\lambda^2 + \lambda^2\mu^2} \tag{22}$$

$$\therefore \lambda^2 \tan A = \mu^2 \tan B = \nu^2 \tan C = \sqrt{(\mu^2\nu^2 + \nu^2\lambda^2 + \lambda^2\mu^2)} \tag{23}$$

also from (10)

$$\frac{\cos A}{a\lambda^2} = \frac{\cos B}{b\mu^2} = \frac{\cos C}{c\nu^2} = \frac{1}{2abc}$$

$$\therefore \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{\sqrt{(\mu^2\nu^2 + \nu^2\lambda^2 + \lambda^2\mu^2)}}{2abc} \tag{24}$$

and if we put

$$S = \frac{a + b + c}{2}$$

we shall find by actual multiplication that

$$\mu^2\nu^2 + \nu^2\lambda^2 + \lambda^2\mu^2 = 16S(S - a)(S - b)(S - c) \tag{25}$$

and the above will agree with the usual formulæ.

On some Theorems relative to Sections of Surfaces of the Second Order.

I.—ON THE SECTIONS OF THE CONE.*

THE plane sections of a surface are given by combining the equation to the surface

$$F\,(xyz) = 0 \tag{1}$$

with the equation to the plane, which will be

$$\lambda x + \mu y + \nu z = 0 \tag{2}$$

or

$$\lambda(x-a) + \mu(y-\beta) + \nu(z-\gamma) = 0 \tag{3}$$

if it passes through a point $\alpha\ \beta\ \gamma$.

Instead however of finding the nature of the section by a transformation of coordinates, a process necessarily long and tedious, we will make use of the distance of a given point from the surface, as that quantity is independent of the coordinate axes.

Let the equations to a line passing through the point $\alpha\ \beta\ \gamma$ be

$$\frac{x-a}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r \tag{4}$$

Then if (4) lies in (3) we have the condition

$$\lambda l + \mu m + \nu n = 0 \tag{5}$$

Combining (4) and (5) with (1) we shall find the equation to the plane section in terms of all the values of r in it.

Before proceeding, we will first determine how curves of the second order, in plano, are distinguished. The general equation of the second order between two variables is

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0 \tag{6}$$

and this represents an

$$\left.\begin{array}{l}\text{ellipse}\\ \text{parabola}\\ \text{hyperbola}\end{array}\right\}\text{according as } B^2 - AC \text{ is}\left\{\begin{array}{l}<0\\ =0\\ >0\end{array}\right.$$

* The first part of this section will be found in Gregory's Solid Geometry; it has been inserted in order to make the latter part intelligible.

and if we substitute for x and y in terms of r from the equations

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = r \tag{7}$$

we find

$$(\mathrm{A}l^2 + 2\mathrm{B}lm + \mathrm{C}m^2)r^2 + \ldots\ldots = 0 \tag{8}$$

and the discriminating condition is equivalent to saying that the curve is an ellipse, parabola, or hyperbola, according as the coefficient of r^2, equated to zero, leads to impossible, equal, or possible values of the ratio $l : m$ or $m : l$.

We shall find this condition equally applicable in three dimensions.

The equation to the cone is

$$\mathrm{P}x^2 + \mathrm{P}'y^2 - \mathrm{P}''z^2 = 0 \tag{9}$$

whence

$$(\mathrm{P}l^2 + \mathrm{P}'m^2 - \mathrm{P}''n^2)r^2 + \ldots\ldots = 0 \tag{10}$$

and the discriminating condition depends upon the nature of the expression

$$\mathrm{P}l^2 + \mathrm{P}'m^2 - \mathrm{P}''n^2 = 0 \tag{11}$$

combined with

$$\lambda l + \mu m + \nu n = 0 \tag{12}$$

eliminating n between these we have

$$(\mathrm{P}\nu^2 - \mathrm{P}''\lambda^2)l^2 + (\mathrm{P}'\nu^2 - \mathrm{P}''\mu^2)m^2 - 2\mathrm{P}''\lambda\mu lm = 0 \tag{13}$$

and the discriminating condition becomes

$$\mathrm{P}''^2\lambda^2\mu^2 - (\mathrm{P}\nu^2 - \mathrm{P}''\lambda^2)(\mathrm{P}'\nu^2 - \mathrm{P}''\mu^2)$$

or

$$-\mathrm{PP}'\nu^2 + \mathrm{P}'\mathrm{P}''\lambda^2\nu^2 + \mathrm{PP}''\nu^2\mu^2$$

or

$$\nu^2(\mathrm{P}'\mathrm{P}''\lambda^2 + \mathrm{P}''\mathrm{P}\mu^2 - \mathrm{PP}'\nu^2) \tag{14}$$

and the nature of the conic section will depend on the sign of the quantity under the bracket.

Suppose the cone to be a right one on a circular base

$$\therefore\ \mathrm{P} = \mathrm{P}' \tag{15}$$

and the condition becomes

$$\nu^2\{\mathrm{PP}''(\lambda^2 + \mu^2) - \mathrm{P}^2\nu^2\}$$

and since

$$\lambda^2 + \mu^2 + \nu^2 = 1 \tag{16}$$

we have

$$\mathrm{P}\nu^2\{\mathrm{P}''(1-\nu^2) - \mathrm{P}\nu^2\}$$

or

$$\mathrm{P}\nu^2\{\mathrm{P}'' - (\mathrm{P} + \mathrm{P}'')\nu^2\}$$

$\therefore$ we have an

$$\left.\begin{array}{l}\text{ellipse}\\ \text{parabola}\\ \text{hyperbola}\end{array}\right\}\text{according as } \nu^2 \text{ is}\left\{\begin{array}{c}>\\=\\<\end{array}\right\}\quad \frac{P''}{P+P''}$$

Now, since the equation to the cone is

$$P(x^2+y^2)-P''z^2=0$$

we have

$$\tan COB = \sqrt{\frac{P''}{P}}$$

where COB = semivertical angle of the cone.

And ν = cosine of $\angle$ between a normal to the plane and the axis of x = sin OED, where OED = angle between the axis of the cone, and the cutting plane.

Hence we have an

$$\left.\begin{array}{l}\text{ellipse}\\ \text{parabola}\\ \text{hyperbola}\end{array}\right\}\text{according as } \sin^2 OED\left\{\begin{array}{c}>\\=\\<\end{array}\right\}\quad \frac{\tan^2 COB}{\sec^2 COB}$$

$$\text{i. e. as } \angle\, OED\left\{\begin{array}{c}>\\=\\<\end{array}\right\}\angle\, COB$$

II.—ON THE PRINCIPAL AXES OF SECTIONS OF SURFACES OF THE SECOND ORDER.*

Let the surface be a central one, then its equation will be

$$Px^2 + P'y^2 + P''z^2 = H \qquad (1)$$

and the equations to a line in the plane of section will be

$$\frac{x-\xi}{l} = \frac{y-\eta}{m} = \frac{z-\zeta}{n} = r \qquad (2)$$

and the condition that it lies in the plane, the direction cosines of a normal to which are

$$\lambda, \quad \mu, \quad \nu,$$

will be

$$\lambda l + \mu m + \nu n = 0 \qquad (3)$$

from (2) we find

$$x = \xi + lr \qquad y = \eta + mr \qquad z = \zeta + nr \qquad (4)$$

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* The following investigations were partly communicated to me by the Rev. B. Price, of Pembroke College.

$\therefore$ substituting in (1)

$$\left.\begin{array}{r} P\xi^2 + P'\eta^2 + P''\zeta^2 + 2(P\xi l + P'\eta m + P''\zeta n)r \\ + (Pl^2 + P'm^2 + P''n^2)r^2 = H \end{array}\right\} \tag{5}$$

then in order to find the principal axes of the section, we must make r pass through the centre, or have the two values of r equal and of opposite signs; and then find r^2 a maximum or minimum. Hence we have

$$Pl\xi + P'm\eta + P''n\zeta = 0 \tag{6}$$

and

$$r^2 = \frac{H - (P\xi^2 + P'\eta^2 + P''\zeta^2)}{Pl^2 + P'm^2 + P''n^2} \tag{7}$$

Now, in order that r may be a maximum or minimum, it will be sufficient that the differential coefficient of the denominator of this fraction be put equal to zero; l, m, n, being subject to the condition

$$l^2 + m^2 + n^2 = 1 \tag{8}$$

Our object now is to eliminate $l\ m\ n$ from the equations (3), (7), and (8). Hence differentiating as above mentioned, we have

$$\left.\begin{array}{r} Pldl + P'mdm + P''ndn = 0 \\ \lambda dl + \mu dm + \nu dn = 0 \\ ldl + mdm + ndn = 0 \end{array}\right\} \tag{9}$$

whence, introducing two indeterminate multipliers σ, τ, we have

$$\left.\begin{array}{r} Pl + \sigma\lambda - \tau l = 0 \\ P'm + \sigma\mu - \tau m = 0 \\ P''n + \sigma\nu - \tau n = 0 \end{array}\right\} \tag{10}$$

multiplying by $l\ m\ n$ respectively, and adding, we find

$$Pl^2 + P'm^2 + P''n^2 = \tau$$

or by (5)

$$\tau = \frac{H - (P\xi^2 + P'\eta^2 + P''\zeta^2)}{r^2} \tag{11}$$

or writing

$$H - (P\xi^2 + P'\eta^2 + P''\zeta^2 = Q \tag{12}$$

hence (10) become

$$\left.\begin{array}{r} \left(P - \frac{Q}{r^2}\right)l + \sigma\lambda = 0 \\ \left(P' - \frac{Q}{r^2}\right)m + \sigma\mu = 0 \\ \left(P'' - \frac{Q}{r^2}\right)n + \sigma\nu = 0 \end{array}\right\} \tag{13}$$

whence

$$-l = \frac{\sigma\lambda}{P - \frac{Q}{r^2}},\ -m = \frac{\sigma\mu}{P' - \frac{Q}{r^2}},\ -n = \frac{\sigma\nu}{P'' - \frac{Q}{r^2}}, \qquad (14)$$

whence, multiplying by λ μ ν respectively, and adding, we find by means of (3)

$$\frac{\lambda^2}{P - \frac{Q}{r^2}} + \frac{\mu^2}{P' - \frac{Q}{r^2}} + \frac{\nu^2}{P'' - \frac{Q}{r^2}} = 0 \qquad (15)$$

from this equation we may easily show that the two greatest and least values of r are at right angles to each other, for let r_1^2 r_2^2 be two roots of this equation, then writing down the equation for each of these values of r, and subtracting, we find,

$$\left(\frac{Q}{r_1^2} - \frac{Q}{r_2^2}\right)\left\{\frac{\lambda^2}{\left(P - \frac{Q}{r_1^2}\right)\left(P - \frac{Q}{r_2^2}\right)} + \frac{\mu^2}{\left(P' - \frac{Q}{r_1^2}\right)\left(P' - \frac{Q}{r_2^2}\right)} + \frac{\nu^2}{\left(P'' - \frac{Q}{r_1^2}\right)\left(P'' - \frac{Q}{r_2^2}\right)}\right\} = 0 \qquad (16)$$

or by (14), if r_1 is different from r_2

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0 \qquad (17)$$

i. e. the two lines are perpendicular.

Also clearing (15) of fractions we have

$$\left(P' - \frac{Q}{r^2}\right)\left(P'' - \frac{Q}{r^2}\right)\lambda^2 + \left(P'' - \frac{Q}{r^2}\right)\left(P - \frac{Q}{r^2}\right)\mu^2 + \left(P - \frac{Q}{r^2}\right)\left(P' - \frac{Q}{r^2}\right)\nu^2 = 0 \quad (18)$$

whence

$$\left.\begin{array}{r}(P'P''\lambda^2 + P''P\mu^2 + PP'\nu^2)r^2 \\ -2Q\{(P' + P'')\lambda^2 + (P'' + P)\mu^2 + (P + P')\nu^2\}r^2 + Q^2 = 0\end{array}\right\} \qquad (19)$$

whence, if a b be the principal semiaxes of the section, we have by the theory of equations

$$a^2 b^2 = \frac{Q^2}{P'P''\lambda^2 + P''P\mu^2 + PP'\nu^2}$$

or

$$ab = \frac{H - (P\xi^2 + P'\eta^2 + P''\zeta^2}{(P'P''\lambda^2 + P''P\mu^2 + PP'\nu^2)^{\frac{1}{2}}} \qquad (20)$$

and

$$a^2 + b^2 = \frac{(P' + P'')\lambda^2 + (P'' + P)\mu^2 + (P' + P)\nu^2}{P'P''\lambda^2 + P''P\mu^2 + PP'\nu^2} \qquad (21)$$

In the case of the cone

$$H = 0 \tag{22}$$

and one of the coefficients as P'' is negative; hence we find real finite values for a and b only when

$$\nu^2 PP' \text{ is } > \lambda^2 P'P'' + \mu^2 P''P$$

i. e. only in the case of the ellipse (by a former paper). In the case of the parabola they become infinite, by the same paper. In that of the hyperbola they or one of them becomes imaginary.

Also the area of an elliptic section

$$\left.\begin{aligned} &= \pi ab \\ &= \frac{\pi}{\sqrt{PP'P''}} \frac{P\xi^2 + P'\eta^2 - P''\zeta^2}{\left\{-\frac{\lambda^2}{P} - \frac{\mu^2}{P'} + \frac{\nu^2}{P''}\right\}^{\frac{1}{2}}} \end{aligned}\right\} \tag{23}$$

III.—If a sphere be inscribed in a cone, and a section of the cone be made by a plane touching the sphere, the point of contact of the plane and sphere will be a focus of the section.

Let the equation to the sphere be

$$x^2 + y^2 + z^2 = r^2 \tag{1}$$

then the curve of contact is given by the equation

$$ax + by + cz = r^2 \tag{2}$$

where $a\ b\ c$ are the coordinates of the vertex of the cone. Let $\xi\ \eta\ \zeta$ be the current coordinates of the cone, then the equation to the generator may be written

$$\frac{\xi - x}{a - x} = \frac{\eta - y}{b - y} = \frac{\zeta - z}{c - z} \tag{3}$$

subtracting (1) from (2) we have

$$x(a-x) + y(b-y) + z(c-z) = 0 \tag{4}$$

whence, by means of (3)

$$x(\xi-x) + y(\eta-y) + z(\zeta-z) = 0 \tag{5}$$

or

$$\xi x + \eta y + \zeta z = r^2 \tag{6}$$

whence (3)

$$= \frac{\xi a + \eta b + \zeta c - r^2}{a^2 + b^2 + c^2 - r^2} = \frac{\xi^2 + \eta^2 + \zeta^2 - r^2}{\xi a + \eta b + \zeta c - r^2} \tag{7}$$

whence

$$(\xi a + \eta b + \zeta c - r^2)^2 = (\xi^2 + \eta^2 + \zeta^2 - r^2)(a^2 + b^2 + c^2 - r^2) \tag{8}$$

which is the equation to the cone enveloping the sphere (1).

Now if we take the plane of the section for that of $\xi\,\eta$

$$\zeta = -r \tag{9}$$

hence the equation to the conic section is

$$\{\xi a + \eta b + r(c-r)\}^2 = (\xi^2 + \eta^2)(a^2 + b^2 + c^2 - r^2) \tag{10}$$

i. e. the radius vector

$$\rho = \sqrt{(\xi^2 + \eta^2)} \tag{11}$$

is a rational function of the coordinates; i. e. the origin is the focus. Similarly the point where the plane of section is touched by another sphere on its other side will be the other focus.

The equation to the enveloping cone may be put in a remarkable form as follows; multiplying out we find

$$\left.\begin{aligned} &\xi^2a^2 + \eta^2b^2 + \zeta^2c^2 - 2r^2(\xi a + \eta b + \zeta c) \\ &\qquad + 2(\eta\xi bc + \zeta\xi ca + \xi\eta ab) \\ &= \xi^2(a^2 + b^2 + c^2 - r^2) + \eta^2(a^2 + b^2 + c^2 - r^2) + \zeta^2(a^2 + b^2 + c^2 - r^2) \\ &\qquad - r^2(a^2 + b^2 + c^2) + r^4 \end{aligned}\right\} \tag{12}$$

or

$$\left.\begin{aligned} &(b^2 + c^2 - r^2)\xi^2 + (c^2 + a^2 - r^2)\eta^2 + (a^2 + b^2 - r^2)\zeta^2 \\ &- 2(bc\zeta\eta + ca\zeta\xi + ab\xi\eta) + 2r^2(a\xi + b\eta + c\zeta) - r^2(a^2 + b^2 + c^2) = 0 \end{aligned}\right\} \tag{13}$$

or

$$\frac{(c\eta - b\zeta)^2}{r^2} + \frac{(a\zeta - c\xi)^2}{r^2} + \frac{(b\xi - a\eta)^2}{r^2} - [(\xi-a)^2 + (\eta-b)^2 + (\zeta-c)^2] = 0 \tag{14}$$

a convenient form for finding the distance of any point on the cone from the vertex.

On the Reduction of the general Equation of the Second Order.

I.—ON SURFACES OF THE SECOND ORDER.

THE general equation of the second order between three variables is

$$Ax^2 + A'y^2 + A''z^2 + 2Byz + 2B'zx + 2B''xy + Cx + C'y + C''z + E = 0 \quad (1)$$

but since this in its present form contains so many terms, it will be worth while to investigate the possibility of simplifying the expression by a change of coordinates.

Let the direction of the coordinate axes be changed by the formulæ

$$x = a\xi + a'\eta + a''\zeta \qquad y = b\xi + b'\eta + b''\zeta \qquad z = c\xi + c'\eta + c''\zeta \quad (2)$$

if then the system be still supposed rectangular, the nine direction cosines will be subject to the six conditions

$$a^2 + a'^2 + a''^2 = 1 \qquad b^2 + b'^2 + b''^2 = 1 \qquad c^2 + c'^2 + c''^2 = 1 \quad (3)$$

$$bc + b'c' + b''c'' = 0 \qquad ca + c'a' + c''a'' = 0 \qquad ab + a'b' + a''b'' = 0 \quad (4)$$

whence also the inverse system

$$a^2 + b^2 + c^2 = 1 \qquad a'^2 + b'^2 + c'^2 = 1 \qquad a''^2 + b''^2 + c''^2 = 1 \quad (5)$$

$$a'a'' + b'b'' + c'c'' = 0 \qquad a''a + b''b + c''c = 0 \qquad aa' + bb' + cc' = 0 \quad (6)$$

may be easily deduced.

On the introduction of the new variables $\xi\ \eta\ \zeta$ by means of (2), the equation (1) takes the form

$$\mathfrak{A}\xi^2 + \mathfrak{A}'\eta^2 + \mathfrak{A}''\zeta^2 + 2\mathfrak{B}\eta\zeta + 2\mathfrak{B}'\zeta\xi + 2\mathfrak{B}''\xi\eta + \mathfrak{C}\xi + \mathfrak{C}'\eta + \mathfrak{C}''\zeta + \mathfrak{E} = 0 \quad (7)$$

where

$$\mathfrak{A} = Aa^2 + A'b^2 + A''c^2 + 2Bbc + 2B'ca + 2B''ab, \quad \mathfrak{A}' = \ldots., \quad \mathfrak{A}'' = \ldots. \quad (8)$$

$$\left.\begin{aligned} \mathfrak{B} &= Aa'a'' + A'b'b'' + A''c'c'' + B(b'c'' + b''c') + B'(c'a'' + c''a') + B''(a'b'' + a''b') \\ \mathfrak{B}' &= Aa''a + A'b''b + A''c''c + B(b''c + bc'') + B'(c''a + ca'') + B''(a''b + ab'') \\ \mathfrak{B}'' &= Aaa' + A'bb' + A''cc' + B(bc' + b'c) + B'(ca' + c'a) + B''(ab' + a'b) \end{aligned}\right\} \quad (9)$$

$$\mathfrak{C} = Ca + C'b + C''c, \quad \mathfrak{C}' = Ca' + C'b' + C''c', \quad \mathfrak{C}'' = Ca'' + C'b'' + C''c'' \quad (10)$$

Since however the nine quantities

$$a, b, c, \qquad a', b', c', \qquad a'', b'', c'',$$

are subject to only the six relations (3) and (4), or (5) and (6), it is allowable to assume three more conditions respecting them; let those be

$$\mathfrak{B} = 0 \qquad \mathfrak{B}' = 0 \qquad \mathfrak{B}'' = 0 \quad (11)$$

When these relations are satisfied the coordinate axes will be called *principal axes.*

It now remains to be proved that when the equations (11) are satisfied the quantities $\mathfrak{A}'\mathfrak{A}'\mathfrak{A}''$ are real, i. e. that the conditions (11) are possible; in order to do this, let the last two of (11) be written thus

$$\left.\begin{array}{l}(\mathrm{A}a+\mathrm{B}'c+\mathrm{B}''b)a' + (\mathrm{A}'b+\mathrm{B}''a+\mathrm{B}c)b' + (\mathrm{A}''c+\mathrm{B}b+\mathrm{B}'a)c'=0\\(\mathrm{A}a+\mathrm{B}'c+\mathrm{B}''b)a'' + (\mathrm{A}'b+\mathrm{B}''a+\mathrm{B}c)b'' + (\mathrm{A}''c+\mathrm{B}b+\mathrm{B}'a)c''=0\end{array}\right\} \quad (12)$$

whence, by symmetrical eliminations,

$$\frac{\mathrm{A}a+\mathrm{B}'c+\mathrm{B}''b}{b'c''-b''c'} = \frac{\mathrm{A}'b+\mathrm{B}''a+\mathrm{B}c}{c'a''-c''a'} = \frac{\mathrm{A}''c+\mathrm{B}b+\mathrm{B}'a}{a'b''-a''b'} \quad (13)$$

But by similar eliminations from the last two of (4) or of (6) there result

$$\frac{b'c''-b''c'}{a} = \frac{c'a''-c''a'}{b} = \frac{a'b''-a''b'}{c} \quad (14)$$

hence (13) may be written

$$\frac{\mathrm{A}a+\mathrm{B}'c+\mathrm{B}''b}{a} = \frac{\mathrm{A}'b+\mathrm{B}''a+\mathrm{B}c}{b} = \frac{\mathrm{A}''c+\mathrm{B}b+\mathrm{B}'a}{c} = \mathfrak{A} \quad (15)$$

with similar expressions for $\mathfrak{A}'\mathfrak{A}''$. Now (15) may also be written

$$\left.\begin{array}{r}(\mathrm{A}-\mathfrak{A})a+\mathrm{B}''b+\mathrm{B}'c=0\\ \mathrm{B}''a+(\mathrm{A}'-\mathfrak{A})b+\mathrm{B}c=0\\ \mathrm{B}'a+\mathrm{B}b+(\mathrm{A}''-\mathfrak{A})c=0\end{array}\right\} \quad (16)$$

whence, by cross multiplication,

$$(\mathfrak{A}-\mathrm{A})(\mathfrak{A}-\mathrm{A}')(\mathfrak{A}-\mathrm{A}'')-\mathrm{B}^2(\mathfrak{A}-\mathrm{A})-\mathrm{B}'^2(\mathfrak{A}-\mathrm{A}')-\mathrm{B}''^2(\mathfrak{A}-\mathrm{A}'')-2\mathrm{B}\mathrm{B}'\mathrm{B}''=0 \quad (17)$$

The same equation would evidently have been deduced if instead of (15) we had taken the analogous system involving $a'b'c'\mathfrak{A}'$, or that involving $a''b''c''\mathfrak{A}''$. It may thus be shown that the three roots of (17) are all real; the equation (17) and the two others, which involve $\mathfrak{A}'$, $\mathfrak{A}''$ respectively, being all cubics, must each have one real root at least.

If however any two of these roots were equal, there would result

$$\left.\begin{array}{r}(\mathrm{A}-\mathfrak{A})a'+\mathrm{B}''b'+\mathrm{B}'c'=0\\(\mathrm{A}-\mathfrak{A})a''+\mathrm{B}''b''+\mathrm{B}'c''=0\end{array}\right\} \quad (18)$$

whence by means of (14) there may be deduced the system

$$\frac{\mathrm{A}-\mathfrak{A}}{a} = \frac{\mathrm{B}''}{b} = \frac{\mathrm{B}'}{c} \quad (19)$$

and similarly, by the assistance of (4) or (6), the analogous systems

$$\frac{\mathrm{B}''}{a'} = \frac{\mathrm{A}'-\mathfrak{A}}{b'} = \frac{\mathrm{B}}{c'} \qquad \frac{\mathrm{B}'}{a''} = \frac{\mathrm{B}}{b''} = \frac{\mathrm{A}''-\mathfrak{A}}{c''} \quad (20)$$

whence by means of (6)

$$B''(A-\mathfrak{A}) + B''(A'-\mathfrak{A}) + BB' = 0$$

and similarly for the other quantities; whence may be deduced the system

$$\left.\begin{aligned} \frac{B'B''}{B} + A' + A'' - 2\mathfrak{A} = 0 \\ \frac{B''B}{B'} + A'' + A - 2\mathfrak{A} = 0 \\ \frac{BB'}{B''} + A + A' - 2\mathfrak{A} = 0 \end{aligned}\right\} \tag{21}$$

or, as they are usually written,

$$A - \frac{B'B''}{B} = A' - \frac{B''B}{B'} = A'' - \frac{BB'}{B''} = A + A' + A'' - 2\mathfrak{A} \tag{22}$$

Consequently, unless the conditions (22) are satisfied, no two of the roots are equal to one another; but whatever is a root of one of the cubics is a root of the other two; hence the equation (17) has in general three real unequal roots.

If none of the quantities $\mathfrak{A}\mathfrak{A}'\mathfrak{A}''$ vanish, it will evidently always be possible to destroy the coefficients of the first powers of the variables by means of removing the origin, without again changing the direction of the coordinate axes. The equation (1) will then be finally reduced to the form

$$P\xi^2 + Q\eta^2 + R\zeta^2 = H \tag{23}$$

It may be observed that in this form of the equation no change is effected by altering the sign of any one or more of the variables $\xi\ \eta\ \zeta$; from this property the present position of the origin is called the centre of the surface. If however any one of the quantities $\mathfrak{A}\ \mathfrak{A}'\ \mathfrak{A}''$ vanish, the last transformation is evidently impracticable; the condition that this may be the case will be derived from equating $\mathfrak{A}$ to zero in (17); there results

$$AA'A'' - AB^2 - A'B'^2 - A''B''^2 + 2BB'B'' = 0 \tag{24}$$

this is, therefore, the condition that the surface may be one without a centre.

It was shown above that (22) were the conditions that (17) should have two equal roots; when this is the case it is evident that the values of $a'b'c'$ would be equal to those of $a\ b\ c$ respectively; but as this would not satisfy (6), the values must be imaginary, and their ratios equal to the ratios of the values of $a\ b\ c$.

Hence the result will be of the form

$$\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c} = \mathrm{k}\sqrt{-} \tag{25}$$

but since by (4) the axis $a'b'c$ must be perpendicular to the plane of the axes of abc and $a''b''c''$, the relations (25) represent that, if the axis $a'b'c'$ were drawn in a plane perpendicular to its proper plane, it would coincide with that of abc, which is the usual interpretation of the symbol $\sqrt{-}$. It may also be remarked that the first two of (6) will then be reduced to one equation only, and consequently the plane of the axis $a''b''c''$ being determined, the direction of it in that plane will be indeterminate; i. e. it may lie in any direction in that plane. The surface will in this case be one of *revolution*, i. e. it may be generated by the revolution of a plane curve round the axis whose direction is determinate.

The conditions that (1) may represent a surface of revolution may also be deduced in the following manner.

Adding and subtracting the three quantities

$$\frac{\mathrm{B'B''}}{\mathrm{B}}x^2 \qquad \frac{\mathrm{B''B}}{\mathrm{B'}}y^2 \qquad \frac{\mathrm{BB'}}{\mathrm{B''}}z^2$$

(1) takes the form

$$\left.\begin{array}{l} \left(\mathrm{A} - \frac{\mathrm{B'B''}}{\mathrm{B}}\right)x^2 + \left(\mathrm{A'} - \frac{\mathrm{B''B}}{\mathrm{B'}}\right)y^2 + \left(\mathrm{A''} - \frac{\mathrm{BB'}}{\mathrm{B''}}\right)z^2 \\ + \mathrm{BB'B''}\left(\frac{x}{\mathrm{B}} + \frac{y}{\mathrm{B'}} + \frac{z}{\mathrm{B''}}\right)^2 + \mathrm{C}x + \mathrm{C'}y + \mathrm{C''}z + \mathrm{E} = 0 \end{array}\right\} \tag{26}$$

Now, when the surface is cut by a plane perpendicular to the axis of revolution a circle is formed, and a sphere may always be drawn through this curve; hence, when the coordinates of the surface satisfy the equation to the plane they must also satisfy the equation to a sphere. The conditions that this may take place are evidently

$$\frac{x}{\mathrm{B}} + \frac{y}{\mathrm{B'}} + \frac{\cancel{x}\,z}{\mathrm{B''}} = 0 \tag{27}$$

and

$$\mathrm{A} - \frac{\mathrm{B'B''}}{\mathrm{B}} = \mathrm{A'} - \frac{\mathrm{B''B}}{\mathrm{B'}} = \mathrm{A''} - \frac{\mathrm{BB'}}{\mathrm{B''}} \tag{28}$$

hence (13) is the equation to a plane perpendicular to the axis of revolution, and (14) are the general conditions for a surface of revolution.

It remains for us to determine the position of the principal axes relatively to the surface; for this purpose let

$$\frac{\mathrm{x}-\alpha}{a}=\frac{\mathrm{y}-\beta}{b}=\frac{\mathrm{z}-\gamma}{c} \tag{29}$$

be the equations to a principal axis; the substitution of these in the numerators of (15) gives

$$\left.\begin{aligned}&\frac{\mathrm{A}(\mathrm{x}-\alpha)+\mathrm{B}''(\mathrm{y}-\beta)+\mathrm{B}'(\mathrm{z}-\gamma)}{a}\\&=\frac{\mathrm{B}''(\mathrm{x}-\alpha)+\mathrm{A}'(\mathrm{y}-\beta)+\mathrm{B}(\mathrm{z}-\gamma)}{b}\\&=\frac{\mathrm{B}'(\mathrm{x}-\alpha)+\mathrm{B}(\mathrm{y}-\beta)+\mathrm{A}''(\mathrm{z}-\gamma)}{c}\end{aligned}\right\} \tag{30}$$

which may also be written thus

$$\frac{\mathrm{Ax}+\mathrm{B}''\mathrm{y}+\mathrm{B}'\mathrm{z}+\mathrm{C}}{a}=\frac{\mathrm{B}''\mathrm{x}+\mathrm{A}'\mathrm{y}+\mathrm{Bz}+\mathrm{C}'}{b}=\frac{\mathrm{B}'\mathrm{x}+\mathrm{By}+\mathrm{A}''\mathrm{z}+\mathrm{C}''}{c} \tag{31}$$

if α, β, γ be so chosen that

$$\left.\begin{aligned}\mathrm{A}\alpha+\mathrm{B}''\beta+\mathrm{B}'\gamma+\mathrm{C}&=0\\\mathrm{B}''\alpha+\mathrm{A}'\beta+\mathrm{B}\gamma+\mathrm{C}'&=0\\\mathrm{B}'\alpha+\mathrm{B}\beta+\mathrm{A}''\gamma+\mathrm{C}''&=0\end{aligned}\right\} \tag{32}$$

From these equations it is easily seen that α, β, γ are the coordinates of the centre of the surface when referred to the original axes. It is also observable that in the values of α, β, γ, deduced from these equations, the common denominator is

$$\mathrm{AA'A''}-\mathrm{AB}^2-\mathrm{A'B'}^2-\mathrm{A''B''}^2+2\mathrm{BB'B''}$$

and consequently, if the centre of the surface be removed to an infinite distance, this quantity must vanish, which in fact agrees with the condition (24). Hence the principal axes pass through the centre of the surface. Moreover, the equation (1) may be written

$$(\mathrm{A}x+\mathrm{B}''y+\mathrm{B}'z+\mathrm{C})x+(\mathrm{B}''x+\mathrm{A}'y+\mathrm{B}z+\mathrm{C}')y+(\mathrm{B}'x+\mathrm{B}y+\mathrm{A}''z+\mathrm{C}'')+\mathrm{E}=0 \tag{33}$$

and the surface may consequently be considered as the envelope of a system of a planes, the direction cosines of whose normals are subject to the conditions (31). Hence when

$$\mathrm{x}=x \qquad \mathrm{y}=y \qquad \mathrm{z}=z \tag{34}$$

i. e. when a principal axis meets the surface, it will be a normal.

Hence there are generally in surfaces of the second order three principal axes passing through the centre of the surface at right angles to one another, and coinciding with the normals at the points in which they meet the surface.

II.—ON PLANE CURVES OF THE SECOND ORDER.

The above method has the advantage of being immediately applicable to the equation to plane curves of the second order.

In this case (1) becomes

$$\mathrm{A}x^2 + \mathrm{A}'y^2 + 2\mathrm{B}xy + \mathrm{C}x + \mathrm{C}'y + \mathrm{E} = 0 \tag{1}$$

the transformation of the coordinates is to be effected by means of the two equations

$$x = a\xi + a'\eta \qquad y = b\xi + b'\eta \tag{2}$$

and the condition that the new axes shall be rectangular will be

$$\left.\begin{array}{c} a^2 + a'^2 = 1 \qquad b^2 + b'^2 = 1 \\ aa + a'b' = 0 \end{array}\right\} \tag{3}$$

which, as in the case of three dimensions, may be transformed into the inverse system

$$\left.\begin{array}{c} a^2 + b^2 = 1 \qquad a'^2 + b'^2 = 1 \\ aa' + bb' = 0 \end{array}\right\} \tag{4}$$

the new coefficients will be given by the equations

$$\left.\begin{array}{c} \mathfrak{A} = \mathrm{A}a^2 + \mathrm{A}'b^2 + 2\mathrm{B}ab \\ \mathfrak{A}' = \mathrm{A}a'^2 + \mathrm{A}'b'^2 + 2\mathrm{B}a'b' \\ \mathfrak{B} = \mathrm{A}aa' + \mathrm{A}'bb' + \mathrm{B}(ab'+a'b) \\ \mathfrak{C} = \mathrm{C}a + \mathrm{C}'b \qquad \mathfrak{C}' = \mathrm{C}a' + \mathrm{C}'b' \end{array}\right\} \tag{5}$$

In order then to destroy the coefficient of the product of the variables we must, as before, have

$$\mathfrak{B} = 0 \tag{6}$$

which by means of (4) and (5) will give the relations

$$\left.\begin{array}{c} (\mathrm{A}a + \mathrm{B}b)a' + (\mathrm{A}'b + \mathrm{B}a)b') = 0 \\ aa' + bb' = 0 \end{array}\right\} \tag{7}$$

the result of which is

$$\frac{\mathrm{A}a + \mathrm{B}b}{a} = \frac{\mathrm{A}'b + \mathrm{B}a}{b} = \mathfrak{A} \tag{8}$$

which may also be written in the form

$$\left.\begin{array}{c} (\mathrm{A} - \mathfrak{A})a + \mathrm{B}b = 0 \\ \mathrm{B}a + (\mathrm{A}' - \mathfrak{A}) = 0 \end{array}\right\} \tag{9}$$

the elimination of a and b from which produces the quadratic

$$(\mathrm{A} - \mathfrak{A})(\mathrm{A}' - \mathfrak{A}) - \mathrm{B}^2 = 0 \tag{10}$$

The condition that the curve may not have a centre will be deducible from (8) in the same way as in the case of surfaces, by equating to zero the coefficient of the square of one of the new variables; this will give

$$Aa + Bb = 0 \qquad A'b + Ba = 0 \tag{11}$$

the elimination of a and b from which produces the relation

$$AA' - B^2 = 0 \tag{12}$$

If the two roots of (10) were equal there would result the relations

$$\frac{\mathfrak{A} - A}{b} = \frac{B}{a} \qquad \frac{B}{b'} = \frac{\mathfrak{A} - A'}{a'} \tag{13}$$

which by means of (4) gives

$$B(\mathfrak{A} - A) + B(\mathfrak{A} - A') = 0 \tag{14}$$

or combining this with (10)

$$B = 0 \qquad A = A' \tag{15}$$

This condition may be also deduced in the same way as in the case of surfaces. For, if the curve represented by (1) is a circle (which is the case corresponding to surfaces of revolution), on writing (1) in the form

$$(A-B)x^2 + (A'-B)y^2 + B(x+y)^2 + Cx + C'y + E = 0 \tag{16}$$

the conditions in question evidently coincide with (15).

In order to determine the point through which the principal axes are drawn, and the angle at which they cut the curve, let

$$\frac{\mathrm{x} - \alpha}{a} = \frac{\mathrm{y} - \beta}{b} \tag{17}$$

be the equation to any principal axis; the combination of this formula with (8) gives

$$\frac{A(\mathrm{x}-\alpha) + B(\mathrm{y}-\beta)}{a} = \frac{A'(\mathrm{y}-\beta) + B(\mathrm{x}-\alpha)}{b} \tag{18}$$

which may be written in the form

$$\frac{A\mathrm{x} + B\mathrm{y} + C}{a} = \frac{B\mathrm{x} + A'\mathrm{y} + C'}{b} \tag{19}$$

if α and β be chosen so as to satisfy the equations

$$A\alpha + B\beta + C = 0 \qquad B\alpha + A'\beta + C' = 0 \tag{20}$$

i. e. if α and β be the coordinates of the centre of the curve.

Again, writing (1) in the form

$$(Ax + By + C)x + (Bx + A'y + C')y + E = 0 \tag{21}$$

and the curve in question may be considered as the envelope of straight lines drawn perpendicular to a straight line, whose equation is (18); consequently whenever

$$\mathrm{x} = x \qquad \mathrm{y} = y \tag{22}$$

i. e. when x y is a point on the curve, or when a principal axis meets the curve, it is also perpendicular to it.

Hence in plane curves of the second order there are in general two principal axes passing through the centre of the curve, at right angles to one another, and coinciding with the normals at the points where they meet the curve.

It will be worth while here to enter more particularly into the determination of the quantities $a\,b$, $a'b'$, $\alpha\,\beta$. For this purpose, multiplying the equations (9) by b, a respectively and subtracting, the quantity $\mathfrak{A}$ will be eliminated, and there will result

$$\mathrm{B}(b^2-a^2) + (\mathrm{A}-\mathrm{A}')ab = 0 \tag{23}$$

or writing

$$a = \sin\theta \tag{24}$$

and consequently instead of the first equation of (4)

$$\sin^2\theta + \cos^2\theta = 1 \tag{25}$$

(23) may be written

$$\mathrm{B}(\cos^2\theta-\sin^2\theta) + (\mathrm{A}-\mathrm{A}')\sin\theta\cos\theta = 0 \tag{26}$$

whence

$$\tan 2\theta = \frac{2\mathrm{B}}{\mathrm{A}'-\mathrm{A}} \tag{27}$$

if the conditions (15) are satisfied the angle θ becomes indeterminate, i. e. any rectangular axes are principal axes.

Again, from (20) the coordinates of the centre of the curve are easily deduced; they are

$$\alpha = \frac{\mathrm{BC}'-\mathrm{A}'\mathrm{C}}{\mathrm{AA}'-\mathrm{B}^2} \qquad \beta = \frac{\mathrm{BC}-\mathrm{AC}'}{\mathrm{AA}'-\mathrm{B}^2} \tag{28}$$

and the centre will be removed to an infinite distance if the condition (12) is satisfied.

From equation (10) it is evident that the nature of the roots, and consequently the form of the curve, depends upon the quantity

$$\mathrm{AA}'-\mathrm{B}^2$$

and the curve will be called an ellipse, a parabola, or an hyperbola, according as the quantity is greater, equal, or less than zero.

III.—INVESTIGATION OF THE VARIOUS KINDS OF SURFACES REPRESENTED BY THE GENERAL EQUATION OF THE SECOND DEGREE.

It was shown in § I. that, whenever all the roots of the cubic, formed for finding the coefficients of the squares of the variables, were finite, the coefficients of the products of the variables might always be destroyed; and, moreover, that unless the condition (24) were satisfied, the coefficients of the first powers of the variables might also be made to vanish by a suitable choice of the origin of coordinates. When all these transformations have been effected, the form of the equation to the surface will be the following,

$$Px^2 + Qy^2 + Rz^2 = H \qquad (1)$$

or, writing for convenience,

$$P = \frac{H}{a^2} \qquad Q = \frac{H}{b^2} \qquad R = \frac{H}{c^2} \qquad (2)$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad (3)$$

As however some of the quantities P, Q, R may in certain cases become negative, the most general form of (3) will be

$$\pm\frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 1 \qquad (4)$$

This equation evidently comprises eight others; viz.

One, when of $a^2b^2c^2$ all are positive.
Three, - - two are positive, and one negative.
Three, - - - negative, and one positive.
One, - - all are negative.

These it will be worth while briefly to consider.

(1) Let $a^2b^2c^2$ be all positive; the equation under consideration will then be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad (5)$$

$$\left.\begin{array}{lll} \text{When } y = 0, \; z = 0 & \text{there results} & x = \pm a \\ z = 0, \; x = 0 & & y = \pm b \\ x = 0, \; y = 0 & & z = \pm c \end{array}\right\} \qquad (6)$$

consequently the surface cuts each of the three coordinate axes in two points, whose distances from the origin are determined by equations (6).

The lengths a, b, c are called the principal axes of the surface.

Again, writing (5) in the forms

$$\left.\begin{aligned}
\frac{y^2}{b^2\left(1-\frac{x^2}{a^2}\right)}+\frac{z^2}{c^2\left(1-\frac{x^2}{a^2}\right)}&=1\\
\frac{z^2}{c^2\left(1-\frac{y^2}{b^2}\right)}+\frac{x^2}{a^2\left(1-\frac{y^2}{b^2}\right)}&=1\\
\frac{x^2}{a^2\left(1-\frac{z^2}{c^2}\right)}+\frac{y^2}{b^2\left(1-\frac{z^2}{c^2}\right)}&=1
\end{aligned}\right\} \qquad (7)$$

it is evident that, if the surface be cut by planes parallel to the three coordinate planes respectively, the sections so formed will be all ellipses, having their centres in the axis perpendicular to which the cutting plane is drawn; it is also observable that the principal axes of the ellipses are decreased as the cutting plane recedes from the centre, by quantities varying directly as the square of the distance of that plane from the centre. It is also evident from (7) that, when

$$\left.\begin{array}{lll}
x=\pm a & y=0 & z=0\\
y=\pm b & z=0 & x=0\\
z=\pm c & x=0 & y=0
\end{array}\right\} \qquad (8)$$

and consequently that the surface cuts the coordinate axes only in the points determined by (6) or (8).

Also,

$$\text{if } x>\pm a, \qquad \text{if } y>\pm b, \qquad \text{or if } z>\pm c$$

the curves (7) become imaginary; and consequently the surface in question does not extend beyond the limits

$$x=\pm a \qquad y=\pm b \qquad z=\pm c \qquad (9)$$

this surface is called the *ellipsoid*.

(2) Let one of the quantities a^2, b^2, c^2 be negative; the equation under consideration will then be one of the system

$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad \frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \qquad (10)$$

Taking the first of these equations

$$\left.\begin{array}{ll}
\text{When } y=0,\ z=0, & \text{there results } x^2+a^2=0\\
z=0,\ x=0, & y=\pm b\\
x=0,\ y=0, & z=\pm c
\end{array}\right\} \qquad (11)$$

hence the surface in question never meets the axis of x, but cuts each of the axes of y and z in two points determined by the equations (11).

Again, writing the first equation of the system (10) in the forms

$$\left.\begin{aligned} \frac{y^2}{b^2\left(1+\frac{x^2}{a^2}\right)}+\frac{z^2}{c^2\left(1+\frac{x^2}{a^2}\right)}&=1 \\ \frac{z^2}{c^2\left(1-\frac{y^2}{b^2}\right)}-\frac{x^2}{a^2\left(1-\frac{y^2}{b^2}\right)}&=1 \\ -\frac{x^2}{a^2\left(1-\frac{z^2}{c^2}\right)}+\frac{y^2}{b^2\left(1-\frac{z^2}{c^2}\right)}&=1 \end{aligned}\right\} \tag{12}$$

it is evident that, if the surface be cut by planes parallel to the plane of yz, there will be formed a series of ellipses (as in the case of the ellipsoid) whose magnitude is extensible *ad infinitum*, the smallest being formed in the plane of yz. The sections made by planes parallel to the plane of zx are hyperbolas, the magnitude of whose principal axes decrease as the cutting plane recedes from the origin to a distance $= \pm b$, when they vanish, and again increase *ad infinitum*, as the cutting plane recedes still farther from the origin. It is also observable that the position of the curve traced on the cutting plane is turned through an angle of 90°, as the distance of that plane from the origin passes through either of the values $\pm b$. Precisely similar results will be obtained by considering the sections of the surface made by planes parallel to the plane of xy. This surface is called the *hyperboloid of one sheet*. Since the forms of the surfaces represented by the two remaining equations of the system (10) are similar to that of the one considered, their position only being altered, it will be unnecessary to examine them any further.

(3) If two of the quantities a^2, b^2, c^2 be negative, the equation under consideration will be one of the system

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \qquad -\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \qquad -\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \tag{13}$$

Taking the first of these equations

$$\left.\begin{array}{lll} \text{When } y=0,\ z=0, & \text{there results} & x=\pm a \\ z=0,\ x=0, & & y^2+b^2=0 \\ x=0,\ y=0, & & z^2+c^2=0 \end{array}\right\} \tag{14}$$

hence the surface in question cuts the axis of x in two points at a distance of $= \pm a$ from the origin; but never meets the axes of y or z.

Again, writing the first equation of the system (13) in the forms

$$\left.\begin{aligned} \frac{y^2}{b^2\left(\frac{x^2}{a^2}-1\right)}+\frac{z^2}{c^2\left(\frac{x^2}{a^2}-1\right)} &= 1 \\ -\frac{z^2}{c^2\left(1+\frac{y^2}{b^2}\right)}+\frac{x^2}{a^2\left(1+\frac{y^2}{b^2}\right)} &= 1 \\ \frac{x^2}{a^2\left(1+\frac{z^2}{c^2}\right)}-\frac{y^2}{b^2\left(1+\frac{z^2}{c^2}\right)} &= 1 \end{aligned}\right\} \tag{15}$$

it is evident that, if the surface be cut by planes parallel to the plane of yz, there will be formed a series of ellipses as long as the distance of the cutting plane from the origin is greater than $\pm a$; the ellipse will vanish when $x = \pm a$, and will become imaginary when x is less than $\pm a$. The sections made by planes parallel to the other coordinate planes will be hyperbolas, the magnitudes of whose principal axes are extensible *ad infinitum.* This surface is called the *hyperboloid of two sheets.* In this case (as in that of the hyperboloid of one sheet) it will be unnecessary to examine the remaining equations of the system.

(4) If all the quantities a^2, b^2, c^2 are negative, the locus of the equation will evidently be entirely imaginary.

If the quantity H, in (1), vanishes, the equation to the surface will take the form

$$\pm\frac{x^2}{a^2}\pm\frac{y^2}{b^2}\pm\frac{z^2}{c^2}=0 \tag{16}$$

(1) If all the quantities a^2, b^2, c^2 are positive, the locus of the equation will be entirely imaginary.

(2) If one of the quantities a^2, b^2, c^2 be negative, and the remaining two negative, the equation under consideration will be one of the system

$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0 \qquad \frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=0 \qquad \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0 \tag{17}$$

In all of which the three variables may vanish together, i. e. the equations

$$x=0 \qquad y=0 \qquad z=0 \tag{18}$$

are consistent with (17). Hence the surface passes through its own centre.

Again, writing the first equation of the system (17) in the forms

$$\frac{\frac{y^2}{b^2}}{\frac{x^2}{a^2}} + \frac{\frac{z^2}{c^2}}{\frac{x^2}{a^2}} = 1 \qquad -\frac{\frac{z^2}{c^2}}{\frac{y^2}{b^2}} + \frac{\frac{x^2}{a^2}}{\frac{y^2}{b^2}} = 1 \qquad \frac{\frac{x^2}{a^2}}{\frac{z^2}{c^2}} - \frac{\frac{y^2}{b^2}}{\frac{z^2}{c^2}} = 1 \qquad (19)$$

it is evident that if the surface be cut by planes parallel to the plane of yz, there will be formed a series of ellipses, the magnitudes of whose principal axes, varying directly as the square of the distance of the cutting plane from the origin, may be increased *ad infinitum.* If the surface be cut by planes parallel to the remaining coordinate planes, there will be produced a series of hyperbolas, subject to the same variations as the ellipses on the planes parallel to the plane of yz.

If the cutting plane be in the plane of yz, then

$$x = 0 \text{ and } \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0 \qquad (20)$$

which is satisfied only by

$$y = 0 \qquad z = 0 \qquad (21)$$

consequently the trace of the surface on the plane of yz is a point.

If the cutting plane be in the plane of zx, then

$$y = 0 \text{ and } \frac{x^2}{a^2} - \frac{z^2}{c^2} = 0 \qquad (22)$$

which represents two straight lines mutually intersecting.

If the cutting plane be in the plane of xy, then

$$z = 0 \text{ and } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \qquad (23)$$

which again represents two straight lines mutually intersecting.

This surface is called the *cone.*

If one of the coefficients of the squares of the variables (as that of Q) vanishes, the equation may be reduced to the form

$$\mathrm{P}x^2 + \mathrm{Q}y^2 + \mathrm{R}z = 0 \qquad (24)$$

which may be written, as in former cases,

$$\pm \frac{x^2}{a^2} \pm \frac{y^2}{b^2} = \pm 2\frac{z}{c} \qquad (25)$$

This formula includes the two following ones,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \pm 2\frac{z}{c} \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 2\frac{z}{c} \qquad (26)$$

The sections of this class of surfaces made by planes parallel to the plane of xy are evidently, in the first case ellipses, in the second hyperbolas, which vanish in the plane of xy, and the magnitudes of whose principal axes, varying directly as the distance of the cutting plane from the origin, may be increased *ad infinitum.* The sections made by planes parallel to the two remaining coordinate planes are parabolas, the magnitude of whose parameters, varying directly as the square of the distance of the cutting plane from the origin, may be increased *ad infinitum.*

It is observable that in the second case the direction of the parabolas in the planes parallel to yz and xz are opposed, while in the first case they are the same. The first of these surfaces is called the *elliptic paraboloid*, the second the *hyperbolic paraboloid.*

The remaining forms which the equation to the surface can take are the following,

$$\pm\frac{y^2}{b^2}\pm\frac{z^2}{c^2}=1 \qquad \pm\frac{z^2}{c^2}\pm\frac{x^2}{a^2}=1 \qquad \pm\frac{x^2}{a^2}\pm\frac{y^2}{b^2}=1 \tag{27}$$

$$\pm\frac{y^2}{b^2}\pm\frac{z^2}{c^2}=0 \qquad \pm\frac{z^2}{c^2}\pm\frac{x^2}{a^2}=0 \qquad \pm\frac{x^2}{a^2}\pm\frac{y^2}{b^2}=0 \tag{28}$$

$$\frac{x^2}{a^2}=1 \qquad \frac{y^2}{b^2}=1 \qquad \frac{z^2}{c^2}=1 \tag{29}$$

$$\frac{x^2}{a^2}\pm\frac{z}{c}=0 \qquad \frac{y^2}{b^2}\pm\frac{z}{c}=0 \tag{30}$$

all of which are comprehended under the general term of *cylindrical surfaces.* These, it is clear, being independent of one variable, will retain the same form, whatever value be given to that variable; they may consequently be considered as generated by the motion of a straight line drawn parallel to the axis of that variable whose coefficients vanish in the equation to the surface.

On the Partial Differential Equations of certain Classes of Surfaces.

I.—DEVELOPABLE SURFACES.

Let the equation to the developable surface be

$$L = 0 \tag{1}$$

and the equation to the plane of which it is the envelope

$$ax + by + cz = d \tag{2}$$

where a, b, c are variable parameters.

Differentiating (1) and (2) we have

$$\left.\begin{aligned} Udx + Vdy + Wdz = 0 \\ adx + bdy + cdz = 0 \end{aligned}\right\} \tag{3}$$

whence by means of an indeterminate multiplier

$$\frac{U}{a} = \frac{V}{b} = \frac{W}{c} = \lambda \tag{4}$$

Again, on differentiating the first of (3), there results

$$Ud^2x + Vd^2y + Wd^2z + dxdU + dydV + dzdW = 0 \tag{5}$$

or by means of (4)

$$\lambda(ad^2x + bd^2y + cd^2z) + dxdU + dydV + dzdW = 0 \tag{6}$$

but on differentiating the second of (3), there results

$$ad^2x + bd^2y + cd^2z = 0 \tag{7}$$

Hence

$$dUdx + dVdy + dWdz = 0 \tag{8}$$

combining which with the first of (3), by means of an indeterminate multiplier, we find

$$\frac{dU}{U} = \frac{dV}{V} = \frac{dW}{W} = \mu \tag{9}$$

or

$$\left.\begin{aligned} \mu U = udx + w'dy + v'dz \\ \mu V = w'dx + vdy + u'dz \\ \mu W = v'dx + u'dy + wdz \end{aligned}\right\} \tag{10}$$

Eliminating μ, dx, dy, dz, from these equations with the assistance of the first of (3), we arrive at the usual result, viz.

$$\left.\begin{array}{r} U^2(vw-u'^2) + V^2(wu-v'^2) + W^2(uv-w'^2) \\ + 2VW(v'w'-uu') + 2WU(w'u'-vv') + 2UV(u'v'-ww') = 0 \end{array}\right\} \quad (11)$$

II.—TUBULAR SURFACES.

Let the equation to the tubular surface be

$$L = 0 \quad (1)$$

and the equation to the sphere of which it is the envelope

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2 \quad (2)$$

where a, b, c are variable parameters. Differentiating (1) and (2) we have

$$\left.\begin{array}{r} Udx + Vdy + Wdz = 0 \\ (x-a)dx + (y-b)dy + (z-c)dz = 0 \end{array}\right\} \quad (3)$$

Whence by means of an indeterminate multiplier

$$\frac{x-a}{U} = \frac{y-b}{V} = \frac{z-c}{W} = \lambda = \frac{r}{P} \quad (4)$$

where

$$P^2 = U^2 + V^2 + W^2 \quad (5)$$

hence

$$\left.\begin{array}{l} Pdx = rdU - \dfrac{U}{P}dP \\ Pdy = rdV - \dfrac{V}{P}dP \\ Pdz = rdW - \dfrac{W}{P}dP \end{array}\right\} \quad (6)$$

whence eliminating r, dx, dy, dz, we arrive at the usual result, viz.

$$\left.\begin{array}{l} P^4 \pm rP\{u(V^2+W^2)+v(W^2+U^2)+w(U^2+V^2)-2u'VW-2v'WU-2w'UV\} \\ \quad + r^2\{U^2(vw-u'^2) + V^2(wu-v'^2) + W^2(uv-w'^2) \\ + 2VW(v'w'-uu') + 2WU(w'u'-vv') + 2UV(u'v'-ww')\} = 0 \end{array}\right\} \quad (7)$$

It may here be remarked, that on r becoming infinite this condition coincides with that given in § I. for Developable Surfaces.

On some Theorems relating to the Curvature of Surfaces.

I.—ON LINES OF CURVATURE.

DEF. A line of curvature on any surface is a locus of a series of its consecutive points, such that normals at each point shall meet the normal at the consecutive one.

Let the equation to the surface be

$$L = f(x, y, z) = 0 \tag{1}$$

then, writing for convenience,

$$U = \frac{dL}{dx} \qquad V = \frac{dL}{dy} \qquad W = \frac{dL}{dx} \tag{2}$$

the quantities U, V, W will be proportional to the direction-cosines of a normal at any point (x, y, z). The equations to this normal are

$$\frac{x-\xi}{U} = \frac{y-\eta}{V} = \frac{z-\zeta}{W} = r \tag{3}$$

where r is the distance from the point (x, y, z) on the surface to a point (ξ, η, ζ) on the normal and distinct from (x, y, z). Also, the equations to a normal at any consecutive point on the surface $(x + dx, y + dy, z + dz)$ and meeting the former normal in the point (ξ, η, ζ) are

$$\frac{x+dx-\xi}{U+dU} = \frac{y+dy-\eta}{V+dV} = \frac{z+dz-\zeta}{W+dW} = r + dr \tag{4}$$

whence, multiplying up and subtracting (3) from (4), there result

$$\left.\begin{aligned} dx &= Udr + (r+dr)dU \\ dy &= Vdr + (r+dr)dV \\ dz &= Wdr + (r+dr)dW \end{aligned}\right\} \tag{5}$$

whence, eliminating r and dr by cross-multiplication, there results

$$(VdW-WdV)dx + (WdU-UdW)dy + (UdV-VdU)dz = 0 \tag{6}$$

which, being independent of ξ, η, ζ, will hold good for any point on the surface. Hence (6) is the differential equation of a surface, by the intersection of which with (1) the lines of curvature are formed. It is observable that since

$$\left.\begin{aligned}
d\mathrm{U} &= \frac{d\mathrm{U}}{dx}dx + \frac{d\mathrm{U}}{dy}dy + \frac{d\mathrm{U}}{dz}dz = \frac{d^2\mathrm{L}}{dx^2}dx + \frac{d^2\mathrm{L}}{dydx}dy + \frac{d^2\mathrm{L}}{dzdx}dz \\
d\mathrm{V} &= \frac{d\mathrm{V}}{dx}dx + \frac{d\mathrm{V}}{dy}dy + \frac{d\mathrm{V}}{dz}dz = \frac{d^2\mathrm{L}}{dxdy}dx + \frac{d^2\mathrm{L}}{dy^2}dy + \frac{d^2\mathrm{L}}{dzdy}dz \\
d\mathrm{W} &= \frac{d\mathrm{W}}{dx}dx + \frac{d\mathrm{W}}{dy}dy + \frac{d\mathrm{W}}{dz}dz = \frac{d^2\mathrm{L}}{dxdz}dx + \frac{d^2\mathrm{L}}{dydz}dy + \frac{d^2\mathrm{L}}{dz^2}dz
\end{aligned}\right\} \qquad (7)$$

the equation (6) will be of the second order in the differentials dx, dy, dz; and consequently will in general give two values for the ratios $dx : dy : dz$; in other words, will represent *two distinct lines* traced upon the surface (1).

Again, suppose the equation (6) to have been integrated, and that the result of the integration is

$$\mathfrak{L} = 0 \qquad (8)$$

the differential of which is

$$\mathfrak{U}dx + \mathfrak{V}dy + \mathfrak{W}dz = 0 \qquad (9)$$

where $\mathfrak{U}, \mathfrak{V}, \mathfrak{W}$ are quantities analogous to U, V, W. Combining this with (6) there results the following system,

$$\frac{\mathrm{V}d\mathrm{W} - \mathrm{W}d\mathrm{V}}{\mathfrak{U}} = \frac{\mathrm{W}d\mathrm{U} - \mathrm{U}d\mathrm{W}}{\mathfrak{V}} = \frac{\mathrm{U}d\mathrm{V} - \mathrm{V}d\mathrm{U}}{\mathfrak{W}} \qquad (10)$$

hence, using the factors U, V, W (as in Theorem I. Pref.) there follows

$$\mathrm{U}\mathfrak{U} + \mathrm{V}\mathfrak{V} + \mathrm{W}\mathfrak{W} = 0 \qquad (11)$$

whence the following theorem may be enunciated; *if two surfaces cut one another in their lines of curvature, they cut one another at right angles.*

Again, let three surfaces whose equations are

$$\mathrm{L} = 0 \qquad \mathfrak{L} = 0 \qquad L = 0 \qquad (12)$$

cut one another at right angles; the conditions that this may be the case are

$$\left.\begin{aligned}
\mathfrak{U}U + \mathfrak{V}V + \mathfrak{W}W &= 0 \\
U\mathrm{U} + V\mathrm{V} + W\mathrm{W} &= 0 \\
\mathrm{U}\mathfrak{U} + \mathrm{V}\mathfrak{V} + \mathrm{W}\mathfrak{W} &= 0
\end{aligned}\right\} \qquad (13)$$

where U, V, W are quantities analogous to U, V, W, or $\mathfrak{U}, \mathfrak{V}, \mathfrak{W}$.

But on differentiating (12) we have

$$\left.\begin{aligned}
\mathrm{U}dx + \mathrm{V}dy + \mathrm{W}dz &= 0 \\
\mathfrak{U}dx + \mathfrak{V}dy + \mathfrak{W}dz &= 0 \\
Udx + Vdy + Wdz &= 0
\end{aligned}\right\} \qquad (14)$$

from the first and second of which there result

$$\frac{dx}{\mathrm{V}\mathfrak{W}-\mathfrak{V}\mathrm{W}}=\frac{dy}{\mathrm{W}\mathfrak{U}-\mathfrak{W}\mathrm{U}}=\frac{dz}{\mathrm{U}\mathfrak{V}-\mathfrak{U}\mathrm{V}} \tag{15}$$

and using the factors

$$\mathrm{V}d\mathrm{W}-\mathrm{W}d\mathrm{V}, \quad \mathrm{W}d\mathrm{U}-\mathrm{U}d\mathrm{W}, \quad \mathrm{U}d\mathrm{V}-\mathrm{V}d\mathrm{U}$$

the numerator of the result will be

$$(\mathrm{V}d\mathrm{W}-\mathrm{W}d\mathrm{V})dx+(\mathrm{W}d\mathrm{U}-\mathrm{U}d\mathrm{W})dy+(\mathrm{U}d\mathrm{V}-\mathrm{V}d\mathrm{U})dz \tag{16}$$

and the denominator

$$(\mathrm{U}^2+\mathrm{V}^2+\mathrm{W}^2)(\mathfrak{U}d\mathrm{V}+\mathfrak{V}d\mathrm{V}+\mathfrak{W}d\mathrm{W}) \tag{17}$$

Hence, writing for convenience

$$\mathrm{U}^2+\mathrm{V}^2+\mathrm{W}^2=\mathrm{P}^2 \tag{18}$$

each of the ratios (15) is equal to

$$-\frac{(\mathrm{V}d\mathrm{W}-\mathrm{W}d\mathrm{V})\frac{dx}{\mathrm{P}^2}+(\mathrm{W}d\mathrm{U}-\mathrm{U}d\mathrm{W})\frac{dy}{\mathrm{P}^2}+(\mathrm{U}d\mathrm{V}-\mathrm{V}d\mathrm{U})\frac{dz}{\mathrm{P}^2}}{\mathfrak{U}d\mathrm{U}+\mathfrak{V}d\mathrm{V}+\mathfrak{W}d\mathrm{W}} \tag{19}$$

Again, using the factors

$$\mathfrak{V}d\mathfrak{W}-\mathfrak{W}d\mathfrak{V}, \qquad \mathfrak{W}d\mathfrak{U}-\mathfrak{U}d\mathfrak{W}, \qquad \mathfrak{U}d\mathfrak{V}-\mathfrak{V}d\mathfrak{U}$$

each of the ratios (15) would be equal to

$$-\frac{(\mathfrak{V}d\mathfrak{W}-\mathfrak{W}d\mathfrak{V})\frac{dx}{\mathfrak{P}^2}+(\mathfrak{W}d\mathfrak{U}-\mathfrak{U}d\mathfrak{W})\frac{dy}{\mathfrak{P}^2}+(\mathfrak{U}d\mathfrak{V}-\mathfrak{V}d\mathfrak{U})\frac{dz}{\mathfrak{P}^2}}{\mathrm{U}d\mathfrak{U}+\mathrm{V}d\mathfrak{V}+\mathrm{W}d\mathfrak{W}} \tag{20}$$

But differentiating the first of (13) we find

$$\mathrm{U}d\mathfrak{U}+\mathrm{V}d\mathfrak{V}+\mathrm{W}d\mathfrak{W}+\mathfrak{U}d\mathrm{U}+\mathfrak{V}d\mathrm{V}+\mathfrak{W}d\mathrm{W}=0 \tag{21}$$

Hence, writing

$$\left.\begin{array}{lll}\mathrm{V}d\mathrm{W}-\mathrm{W}d\mathrm{V}=\mathrm{P}^2d\mathrm{A} & \mathrm{W}d\mathrm{U}-\mathrm{U}d\mathrm{W}=\mathrm{P}^2d\mathrm{B} & \mathrm{U}d\mathrm{V}-\mathrm{V}d\mathrm{U}=\mathrm{P}^2d\mathrm{C}\\ \mathfrak{V}d\mathfrak{W}-\mathfrak{W}d\mathfrak{V}=\mathfrak{P}^2d\mathfrak{A} & \mathfrak{W}d\mathfrak{U}-\mathfrak{U}d\mathfrak{W}=\mathfrak{P}^2d\mathfrak{B} & \mathfrak{U}d\mathfrak{V}-\mathfrak{V}d\mathfrak{U}=\mathfrak{P}^2d\mathfrak{C}\\ VdW-WdV=P^2dA & WdU-UdW=P^2dB & UdV-VdU=P^2dC\end{array}\right\} \tag{22}$$

and forming quantities similar to (19) and (20) for U, V, W, we should have the system

$$\left.\begin{array}{r}(d\mathfrak{A}+dA)dx+(d\mathfrak{B}+dB)dy+(d\mathfrak{C}+dC)dz=0\\ (dA+d\mathrm{A})dx+(dB+d\mathrm{B})dy+(dC+d\mathrm{C})dz=0\\ (d\mathrm{A}+d\mathfrak{A})dx+(d\mathrm{B}+d\mathfrak{B})dy+(d\mathrm{C}+d\mathfrak{C})dz=0\end{array}\right\} \tag{23}$$

or

$$\left.\begin{array}{r}d\mathrm{A}dx+d\mathrm{B}dy+d\mathrm{C}dz=0\\ d\mathfrak{A}dx+d\mathfrak{B}dy+d\mathfrak{C}dz=0\\ dAdx+dBdy+dCdz=0\end{array}\right\} \tag{24}$$

which are the equations to the lines of curvature on the surfaces L, $\mathfrak{L}$, L respectively; hence *if three surfaces cut one another at right angles* *the lines of intersection of any one of the surfaces with the other two are its lines of curvature ;* which is Dupin's Theorem.

From this it also follows that *the two lines of curvature on any surface are perpendicular to one another.*

There is one other remarkable property of lines of curvature which it will be worth while to investigate before proceeding further.

Writing for convenience

$$\left.\begin{aligned} H &= u + \frac{U}{VW}(Uu' - Vv' - Ww') \\ K &= v + \frac{V}{WU}(Vv' - Ww' - Uu') \\ L &= w + \frac{W}{UV}(Ww' - Uu' - Vv') \end{aligned}\right\} \tag{25}$$

there results

$$\frac{P}{\rho} = Hl^2 + Km^2 + Ln^2 \tag{26}$$

where ρ is the radius of curvature of the surface, and

$$l = \frac{dx}{ds} \qquad m = \frac{dy}{ds} \qquad n = \frac{dz}{ds} \tag{27}$$

also from the equation to the surface

$$Ul + Vm + Wn = 0 \tag{28}$$

Now in order to find the variations of ρ corresponding to the variations of l, m, n, we have from (26) and (28)

$$Hldl + Kmdm + Lndn = \tfrac{1}{2}d\left(\frac{P}{\rho}\right) \tag{29}$$

and

$$Udl + Vdm + Wdn = 0 \tag{30}$$

but

$$l^2 + m^2 + n^2 = 1 \tag{31}$$

hence

$$ldl + mdm + ndn = 0 \tag{32}$$

But on a line of curvature the relation

$$Hl(Wm - Vn) + Km(Un - Wl) + Ln(Vl - Um) = 0 \tag{33}$$

is satisfied, which gives by cross-multiplication from (29), (30), and (32),

$$d\left(\frac{P}{\rho}\right) = 0 \tag{34}$$

Hence *on a line of curvature the radius of curvature is a maximum or minimum :* or, *the lines of curvature pass through the principal sections of the surface.*

II.—UMBILICI.

Def. An umbilicus is a point on a surface round which (for all points situated indefinitely near the point in question) the curvature is the same in every direction. The directions of the lines of curvature will consequently become indeterminate at an umbilicus.

In order to find the conditions for the existence of such a point, we have

$$\mathrm{V}d\mathrm{W} - \mathrm{W}d\mathrm{V} = (\mathrm{V}v' - \mathrm{W}w')dx + (\mathrm{V}u' - \mathrm{W}v)dy + (\mathrm{V}w - \mathrm{W}u')dz \tag{1}$$

also

$$\mathrm{U}dx + \mathrm{V}dy + \mathrm{W}dz = 0 \tag{2}$$

therefore eliminating dx

$$\mathrm{V}a\mathrm{W} - \mathrm{W}d\mathrm{V} = - \mathrm{WK}dy + \mathrm{VL}dz \tag{3}$$

Modifying the other terms of (6) of § I. in the same way, and arranging the result, the transformed equation becomes

$$\mathrm{U}(\mathrm{K}-\mathrm{L})dydz + \mathrm{V}(\mathrm{L}-\mathrm{H})dzdx + \mathrm{W}(\mathrm{H}-\mathrm{K})dxdy = 0 \tag{4}$$

hence if

$$\mathrm{H} = \mathrm{K} = \mathrm{L} \tag{5}$$

the equation to the lines of curvature is satisfied independently of dx, dy, dz. But since when either U, V, or W vanish these conditions become indeterminate, it will be necessary to make some transformations, in order to put them into a determinate form. Suppose then that $\mathrm{U} = 0$. Multiplying (5) throughout by U V W, and substituting for H, K, L, from (25) of § I., we see that when U vanishes

$$\mathrm{V}v' - \mathrm{W}w' = 0 \tag{6}$$

also writing for convenience

$$\mathrm{V}v' - \mathrm{W}w' = \mathrm{O} \tag{7}$$

(5) becomes

$$u = v + \frac{\mathrm{V}}{\mathrm{UW}}(\mathrm{O} - \mathrm{U}u') = w + \frac{\mathrm{W}}{\mathrm{UV}}(-\mathrm{O} - \mathrm{U}u') \tag{8}$$

Whence, using the factors $\frac{\mathrm{W}}{\mathrm{V}}\, \frac{\mathrm{V}}{\mathrm{W}}$, there results

$$u = \frac{\frac{\mathrm{W}}{\mathrm{V}}v + \frac{\mathrm{V}}{\mathrm{W}}w - 2\frac{\mathrm{U}}{\mathrm{U}}u'}{\frac{\mathrm{W}}{\mathrm{V}} + \frac{\mathrm{V}}{\mathrm{W}}} \tag{9}$$

Hence we obtain the following systems, any one of which are the conditions for the existence of an umbilicus.

$$U = 0 \qquad Vv' - Ww' = 0 \qquad u = \frac{V^2 w - 2VWu' + W^2 v}{V^2 + W^2} \tag{11}$$

$$V = 0 \qquad Ww' - Uu' = 0 \qquad v = \frac{W^2 u - 2\,WUv' + U^2 w}{W^2 + U^2} \tag{12}$$

$$W = 0 \qquad Uu' - Vv' = 0 \qquad w = \frac{U^2 v - 2UVw' + V^2 u}{U^2 + V^2} \tag{13}$$

III.—ON THE LINES OF CURVATURE ON AN ELLIPSOID.

Various methods have been proposed for investigating the nature of the lines of curvature on an ellipsoid; but as all have involved certain transformations more or less laborious in order to integrate the equation, it appeared that these tedious operations might be advantageously avoided by the following course. The equimomental surface was first brought into connexion with the problem by Mr. Cayley, in the Cambridge and Dublin Mathematical Journal, May 1846.

Let the equation to the ellipsoid be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{1}$$

then the equation to the lines of curvature is

$$(b^2 - c^2)\frac{x}{dx} + (c^2 - a^2)\frac{y}{dy} + (a^2 - b^2)\frac{z}{dz} = 0 \tag{2}$$

Now if α, β, γ be so chosen that

$$a^2 - \alpha^2 = b^2 - \beta^2 = c^2 - \gamma^2 = h \tag{3}$$

the equation (2) may be replaced by

$$(\beta^2 - \gamma^2)\frac{x}{dx} + (\gamma^2 - \alpha^2)\frac{y}{dy} + (\alpha^2 - \beta^2)\frac{z}{dz} = 0 \tag{4}$$

But the equation to the ellipsoid whose lines of curvature are given by (4) is

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 1 \tag{5}$$

or substituting from (3)

$$\frac{x^2}{a^2 + h} + \frac{y^2}{b^2 + h} + \frac{z^2}{c^2 + h} = 1 \tag{6}$$

Hence the lines of curvature on the ellipsoid (1) coincide with those on the surface (6), but (4) are the conditions that (6) shall be confocal with (1); consequently, *the lines of curvature on a central surface of the second order are formed by the intersection of that surface with confocal surfaces of the same order.*

Again, if instead of (3) the following system had been employed

$$a^2 - \alpha'^2 = b^2 - \beta'^2 = c^2 - \gamma'^2 = k \tag{7}$$

we should have had instead of (6)

$$\frac{x^2}{a^2 + k} + \frac{y^2}{b^2 + k} + \frac{z^2}{c^2 + k} = 1 \tag{8}$$

Now subtracting (6) from (1), (8) from (1), and (8) from (6), there result

$$\left.\begin{array}{c} \dfrac{x^2}{a^2(a^2+h)} + \dfrac{y^2}{b^2(b^2+h)} + \dfrac{z^2}{c^2(c^2+h)} = 0 \qquad \dfrac{x^2}{a^2(a^2+k)} + \dfrac{y^2}{b^2(b^2+k)} + \dfrac{z^2}{c^2(c^2+k)} = 0 \\ \dfrac{x^2}{(a^2+h)(a^2+k)} + \dfrac{y^2}{(b^2+h)(b^2+k)} + \dfrac{z^2}{(c^2+h)(c^2+k)} = 0 \end{array}\right\} \tag{9}$$

Hence *the lines of curvature on a central surface of the second order are formed by the intersection of that surface with cones of the same order.*

Again, from (9) there result by symmetrical elimination

$$\left.\begin{array}{c} \dfrac{\dfrac{x^2}{a^2}}{(b^2-c^2)(a^2+h)(a^2+k)} = \dfrac{\dfrac{y^2}{b^2}}{(c^2-a^2)(b^2+h)(b^2+k)} = \dfrac{\dfrac{z^2}{c^2}}{(a^2-b^2)(c^2+h)(c^2+k)} \\ = \dfrac{r^2}{(h+k)[a^4(b^2-c^2)+b^4(c^2-a^2)+c^4(a^2-b^2)]+a^6(b^2-c^2)+b^6(c^2-a^2)+c^6(a^2-b^2)} \\ = \dfrac{1}{a^4(b^2-c^2) + b^4(c^2-a^2) + c^4(a^2-b^2)} \end{array}\right\} \tag{10}$$

whence

$$r^2 = h + k + a^2 + b^2 + c^2 \tag{11}$$

and (8) may consequently be written thus,

$$\frac{x^2}{r^2 - h^2 - b^2 - c^2} + \frac{y^2}{r^2 - h - c^2 - a^2} + \frac{z^2}{r^2 - h - a^2 - b^2} = 1 \tag{12}$$

Hence *the lines of curvature on a central surface of the second order are formed by the intersection of that surface with an equimomental surface (which in certain cases takes the form of the wave surface).*

The perpendicularity of the lines of curvature on an ellipsoid follows immediately from (9); those equations being in fact the conditions that the system of surfaces shall be orthogonal.

On certain Formulæ for the Transformation of Coordinates.

Mr. Cayley has, in a paper in the Cambridge Mathematical Journal, quoted some very beautiful formulæ for determining the position of two sets of rectangular axes with respect to each other, employing rational functions of three quantities only. The geometrical method, however, by which he has deduced them is somewhat complex; it is hoped, therefore, that the following analytical investigations will be found as elegant and more simple.

Let the nine direction-cosines be

$$ll'l'', \qquad mm'm'', \qquad nn'n'';$$

then since the two systems are rectangular, they are connected by the following six formulæ of relation,

$$l^2 + m^2 + n^2 = 1 \qquad l'^2 + m'^2 + n'^2 = 1 \qquad l''^2 + m''^2 + n''^2 = 1 \qquad (1)$$

$$l'l'' + m'm'' + n'n'' = 0 \qquad l''l + m''m + n''n = 0 \qquad ll' + mm' + nn' = 0 \qquad (2)$$

whence we easily deduce the inverse systems

$$l^2 + l'^2 + l''^2 = 1 \qquad m^2 + m'^2 + m''^2 = 1 \qquad n^2 + n'^2 + n''^2 = 1 \qquad (3)$$

$$mn + m'n' + m''n'' = 0 \qquad nl + n'l' + n''l'' = 0 \qquad lm + l'm' + l''m'' = 0 \qquad (4)$$

Now subtracting (3) from (1)

$$m^2 + n^2 - l'^2 - l''^2 = l^2 + n'^2 - m^2 - m''^2 = l''^2 + m''^2 - n^2 - n'^2 = 0$$

whence

$$n'^2 - m''^2 = l''^2 - n^2 = m^2 - l'^2 \qquad (5)$$

Let us now assume

$$\frac{\lambda}{n' - m''} = \frac{\mu}{l'' - n} = \frac{\nu}{m - l'} \qquad (6)$$

combining which with (6) we arrive at

$$(n' + m'')\lambda = (l'' + n)\mu = (m + l')\nu$$

or

$$\frac{\mu\nu}{n' + m''} = \frac{\nu\lambda}{l'' + n} = \frac{\lambda\mu}{m + l'} \qquad (7)$$

Now we have introduced three quantities, but have made only two assumptions concerning them; we are therefore at liberty to make another; let this be

$$(6) = (7) \tag{8}$$

Then we easily deduce

$$\frac{\mu\nu+\lambda}{n'} = \frac{\nu\lambda+\mu}{l''} = \frac{\lambda\mu+\nu}{m} = \frac{\mu\nu-\lambda}{m''} = \frac{\nu\lambda-\mu}{n} = \frac{\lambda\mu-\nu}{l'} \tag{9}$$

whence

$$\left.\begin{aligned}
&\frac{l'^2}{(\lambda\mu-\nu)^2} = \frac{l''^2}{(\lambda\nu+\mu)^2} = \frac{1-l^2}{(\lambda\mu-\nu)^2+(\lambda\nu+\mu)^2} = \frac{4(1-l^2)}{(1+\lambda^2+\mu^2+\nu^2)^2-(1+\lambda^2-\mu^2-\nu^2)^2}\\
&= \frac{m''^2}{(\mu\nu-\lambda)^2} = \frac{m^2}{(\lambda\mu+\nu)^2} = \frac{1-m'^2}{(\mu\nu-\lambda)^2+(\mu\lambda+\nu)^2} = \frac{4(1-m'^2)}{(1+\lambda^2+\mu^2+\nu^2)^2-(1+\mu^2-\nu^2-\lambda^2)^2}\\
&= \frac{n^2}{(\nu\lambda-\mu)^2} = \frac{n'^2}{(\nu\mu+\lambda)^2} = \frac{1-n''^2}{(\nu\lambda-\mu)^2+(\nu\mu+\lambda)^2} = \frac{4(1-n''^2)}{(1+\lambda^2+\mu^2+\nu^2)^2-(1+\nu^2-\lambda^2-\mu^2)^2}
\end{aligned}\right\} \tag{10}$$

Now it is obvious that we may from inspection find a solution which will satisfy these equations, but it will be more satisfactory to solve them directly; we shall arrive at the same result.

We will write (10) for convenience thus

$$\frac{l}{\sigma} = \frac{l'}{\alpha'} = \frac{l''}{\alpha''} = \frac{m}{\beta} = \frac{m'}{\tau} = \frac{m''}{\beta''} = \frac{n}{\gamma} = \frac{n'}{\gamma'} = \frac{n''}{\upsilon} = \frac{1}{\kappa} \tag{11}$$

The values of

$$\alpha'\alpha'', \qquad \beta\beta'', \qquad \gamma\gamma'$$

are obvious; our object is to determine

$$\sigma,\ \tau,\ \upsilon,\ \kappa.$$

Now since the two systems are rectangular we have the conditions

$$\left.\begin{aligned}
l'l''+m'm''+n'n''&=0\\
l''l+m''m+n''n&=0\\
ll'+mm'+nn'&=0
\end{aligned}\right\} \tag{12}$$

whence by (11)

$$\left.\begin{aligned}
0.\sigma+\beta''\tau+\gamma'\upsilon &= -\alpha'\alpha''\\
\alpha''\sigma+0.\tau+\gamma\upsilon &= -\beta''\beta\\
\alpha'\sigma+\beta\tau+0\upsilon &= -\gamma\gamma'
\end{aligned}\right\} \tag{13}$$

whence by cross multiplication

$$\left.\begin{aligned}
&\frac{\sigma}{\alpha'\alpha''\beta\gamma-\beta''\gamma'(\beta^2+\gamma^2)} = \frac{\tau}{\beta''\beta\gamma'\alpha'-\gamma\alpha''(\gamma'^2+\alpha'^2)} = \frac{\upsilon}{\gamma\gamma'\alpha''\beta''-\alpha'\beta(\alpha''^2+\beta''^2)}\\
&= \frac{1}{\alpha'\beta''\gamma+\alpha''\beta\gamma'}
\end{aligned}\right\} \tag{14}$$

But

$$\left.\begin{aligned} a'\beta''\gamma + a''\beta\gamma' &= (\mu\nu+\lambda)(\nu\lambda+\mu)(\lambda\mu+\nu) + (\mu\nu-\lambda)(\nu\lambda-\mu)(\lambda\mu-\nu) \\ &= 2(\lambda^2\mu^2\nu^2+\mu^2\nu^2+\nu^2\lambda^2+\lambda^2\mu^2) \end{aligned}\right\} \quad (15)$$

and

$$\left.\begin{aligned} a'a''\beta\gamma-\beta''\gamma'(\beta^2+\gamma^2) &= (\lambda^2\mu^2-\nu^2)(\lambda^2\nu^2-\mu^2)-(\mu^2\nu^2-\lambda^2)(1+\lambda^2)(\mu^2+\nu^2) \\ &= (\lambda^2\mu^2\nu^2+\mu^2\nu^2+\nu^2\lambda^2+\lambda^2\mu^2)(1+\lambda^2-\mu^2-\nu^2) \end{aligned}\right\} \quad (16)$$

whence forming symmetrical quantities for τ and υ, we have

$$\frac{\sigma}{1+\lambda^2-\mu^2-\nu^2} = \frac{\tau}{1+\mu^2-\nu^2-\lambda^2} = \frac{\upsilon}{1+\nu^2-\lambda^2-\mu^2} = \frac{1}{2} \quad (17)$$

whence by (10)

$$\kappa = 1+\lambda^2+\mu^2+\nu^2 \quad (18)$$

whence we immediately arrive at the system given by Mr. Cayley

$$\left.\begin{array}{lll} \kappa l = 1+\lambda^2-\mu^2-\nu^2 & \kappa m = 2(\lambda\mu+\nu) & \kappa n = 2(\nu\lambda-\mu) \\ \kappa l' = 2(\lambda\mu-\nu) & \kappa m' = 1+\mu^2-\nu^2-\lambda^2 & \kappa n' = 2(\nu\mu+\lambda) \\ \kappa l'' = 2(\lambda\nu+\mu) & \kappa m'' = 2(\mu\nu-\lambda) & \kappa n'' = 1+\nu^2-\lambda^2-\mu^2 \end{array}\right\} \quad (19)$$

On the Principle of Virtual Velocities.

By the Rev. B. Price of Pembroke College.

The object of the following paper being to deduce the equation of virtual velocities from more abstract principles than is attempted in the Mécanique Analytique, it may be allowable to make some few remarks on what Lagrange has there given as the proof of the principle, but which appears to the writer of the present paper to be nothing more than an illustration.

Lagrange having acknowledged that the principles of mechanics adopted by the older mathematicians are nothing more than particular expressions of the equation of virtual velocities under different points of view, conceives that the equation depends on a principle different to that of the lever and the composition of forces, viz. on that of the pully; and by means of this constructing and measuring a system of forces, which act on a body or system of connected particles, proceeds to give what has been usually termed by English writers, Lagrange's Proof of the Principle of Virtual Velocities.

It seems, however, to the writer of the present paper, that no advantage is gained by the introduction of the pully. The pully is not nearer to the first principles of mechanics than the lever, the inclined plane, or the parallelogram of forces; whereby Lagrange, in this proof, is not nearer to mechanical axioms than was Archimedes or Varignon. Reactions of various kinds, tensions, rigidity of cord, friction, and so on, are forces introduced in the system of pullies and blocks of which no account is taken; and no principles or axioms are previously laid down which authorize us to neglect these. Lagrange seems to make the principle, which is nothing short of every problem in mechanics, to depend on a single problem, viz. that of the pully and the block. He deduces the general type of every mechanical equation from an equation which he, in a manner, proves to be true in the case of forces acting through a particular combination of the pully. It seems to the writer of the present paper, that if Lagrange intended his problem to be a proof of the principle, he is guilty of the fallacy of concluding the general equation from the particular instance.

In the method pursued in the following pages the three laws of motion are assumed as axioms, and upon them the reasoning is founded. It is conceived, however, that they are not empirical laws, but general truths, adapted to the principles of mechanics, which we may thus state :—

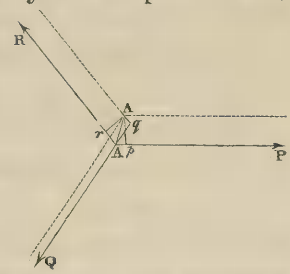
I. " There is no effect without a cause ;" which may be thus adapted, " a particle or body at rest remains at rest, and a particle or body in " motion continues to move uniformly in a straight line, unless acted " on by some force external to itself."

II. " Every cause produces its own effect ;" and we may thus state the adapted mechanical law, " a force acting on a body under the " action of other forces produces its own effect equally as if the other " forces did not act."

III. " Action is accompanied by an equal and opposite reaction ;" that is, " whenever a mechanical force acts there is a reaction simultaneous, " equal, and opposite." In what manner action is to be estimated must be derived from experience ; and on this subject more will be said below. It may be also worth while to notice that the methods of measuring velocity and accelerating force by means of differentials (not differential coefficients) are considered to be known ; that D'Alembert's principle is assumed as axiomatic ; and that wherever the differential calculus is introduced it is considered to be a calculus of differentials or infinitesimals. But before proceeding it is necessary distinctly to understand what is meant by virtual velocity, in what manner virtual velocities are to be estimated, with what signs to be affected, and then formally to enunciate the principle.

I. Meaning of the expression " Virtual Velocity."

By virtual velocity of a point or of a particle is meant, that velocity with which the point or the particle would begin to move supposing it to be disturbed from its position ; or it is that velocity with which, at the first instant of its motion, any point or particle moves, supposing such a motion to be possible. Hence, when a number of forces act on a body or a system of points connected in any manner so as to produce equilibrium, were a slight motion consistently with the geometrical relations of the points to be given to the system, the small spaces described by the point of application of each force relatively to the direction of the force in a very short time δt is a correct measure of the virtual velocity of the point, that is of the force, inasmuch as $\frac{\delta s}{\delta t}$ is the velocity of the point; and if δt be the same for all the points, the virtual velocity must vary as δs; and relatively to this direction of the force, because the force can produce motion or pressure only in its own direction, and therefore whatever small motion the point of application of a force may have in any other direction, that is due to some other force or forces; as, for instance, suppose P, Q, R to be three forces acting on a particle A, and to be so related in intensity and direction as to keep it at rest; conceive the point A to be moved over a very small space to A′, the absolute distance over which A has moved is AA′; but the virtual velocities of the forces are proportional to the several projections of AA′ on the several directions of the forces: viz. the virtual velocity of P is measured by Ap, the virtual velocity of Q by Aq, and of R by Ar; the displacement being so very small that we may consider the directions of the forces still to be parallel to their former directions in the undisturbed system; and the virtual velocity is to be considered positive when the projection of the absolute space over which the point of application of the force is moved is towards the point whence the force acts, as is the case in P and R, and negative when it falls in the direction produced backwards, as in Q.*



* As the parallelogram of forces is with great facility deduced from the diagram given in the text, it may be worth while to anticipate the equation of virtual velocities for the purpose of exhibiting the several steps.

Let $A'AP = \theta$, $AA' = \delta s$, $RAQ = \alpha$, $PAR = \beta$, $QAP = \gamma$, the equation of virtual velocities is in this case $P \, . \, Ap - Q \, . \, Aq + R \, . \, Ar = 0$ and substituting for Ap, Aq, Ar their values, we have

$$P \, . \, \delta s \cos \theta + Q \, . \, \delta s \cos (\theta + \gamma) + R \, . \, \delta s \cos (\beta - \theta) = 0$$

dividing this by δs, expanding the cosines of the multiple arcs, &c. we have

$$P + Q \cos \gamma + R \cos \beta + (R \sin \beta - Q \sin \gamma) \tan \theta = 0$$

whence as θ is indeterminate

$$P + Q \cos \gamma = - R \cos \beta \qquad (1)$$

$$Q \sin \gamma = R \sin \beta \qquad (2)$$

Squaring (1) and (2), and adding $R^2 = P^2 + Q^2 + 2PQ \cos \gamma$, the parallelogram of forces, and from (2) $\frac{P}{\sin \alpha} = \frac{Q}{\sin \beta} = \frac{R}{\sin \gamma}$ from the symmetry, the triangle of forces.

II. Enunciation of the principle of Virtual Velocities.

The principal of virtual velocities is this,—when any number of forces are so combined in both intensity and direction as to produce equilibrium, the sum of the products of each force and its virtual velocity is equal to zero; or if any number of forces acting on a body or system of particles are in equilibrium, the forces are inversely proportionate to their virtual velocities; whence if P be the general type of a force, and δp the general type of the small distance relatively to its own direction over which the point of application of P moves, owing to a slight displacement of the system, then

$$\Sigma . P\delta p = 0$$

and we may thus formally state the theorem:

If any system of points or of bodies acted on by any number of forces is in equilibrium, and if a small motion is given to the system consistently with its geometrical conditions, by means of which the points of application of the forces move through small spaces, these spaces relatively to the directions of the forces are proportional to the virtual velocities, and the algebraical sum of the product of each force and its virtual velocity is equal to zero; the velocities being considered positive when the spaces described are towards the points whence the forces act, and negative when in direction of the force produced backwards.

III. Explanation of the Principle.

This principle involves two considerations,

1st. In what manner are forces producing pressure or motion to be estimated? How are their effects to be measured?

2nd. What do we mean by equilibrium; that is, in what particular forms and in what proportion, as well as to intensity as to direction, are forces to be combined to produce equilibrium?

As to the first, daily experience shows us that the effect of a force depends on two circumstances, the mass of the moving body which contains and is acted on by the force, and the velocity with which it moves; the effect we perceive to be greater the greater the moving mass, and the greater the velocity; whence, in mathematical language, we say the effect is a function of the mass and the velocity; but what function? What we may call the "proof experiment" of Attwood shows that pressure producing motion or moving force is to be measured by the product of the mass moved and the accelerating force; that is, in order to estimate the statical effect of a moving mass, we must multiply the abstract accelerating force by the concrete mass ; or, as we may state the case under another point of view, in order to impress on a given mass a certain velocity in a certain direction (*i. e.* in order to produce a certain effect) a force must be applied, the intensity of which must be proportional to the product of the mass to be moved and the velocity with which it is to move, and the direction of course the same as that in which the body is to move ; this product of the mass and the velocity is called the quantity of motion. Or again, supposing a body to contain P particles, each equal to m, m being the unit particle, and suppose the body to move in a given direction with a certain velocity measured by p, the distance over which it passes in a short time, then the effect of each particle is equal to mp, and therefore the effect of the body being the aggregate of all the effects of the several constituent particles is equal to Pp; for although it may appear that some of the effective forces, that is, some of the motion, is lost, owing to the mutual connexion of the particles, as for instance the molecular forces, yet by the principle of D'Alembert, as long as the body is rigid, and there is no motion of the particles relatively to one another, these forces neutralize each other, and the body notwithstanding produces a force of precisely the same intensity as the sum of the forces due to each particle m, were each to move independent of and separate from the others ; whence then it appears that the correct measure of the effect of a force arising from a moving body is the product of the mass moved and the velocity with which it moves ; and this being so,

The second consideration comes in ; in what manner must forces measured as to their effects in the way we have just explained be combined so that a body under the action of them may be at rest ? Forces are, I say, in equilibrium, when the absolute sum of their effects measured as above is less than in any other state, when the system is slightly deranged :—for conceive a system of bodies or of points which is in equilibrium to be slightly disturbed by the introduction of a new force external to the system, the equilibrium of the former system is thereby destroyed, but the new system will arrange itself in a position of equilibrium, and the resultant of all the forces in the former system will be such in intensity and direction as exactly to counterbalance the force which has been introduced ; thus the sum of all the forces of the deranged system is greater than the sum under the former balanced system by twice the resultant, that is by twice the force which we have introduced in deranging the system; whence it appears that if the body or the system be at rest, the sum of the forces is less than if it be disturbed, and therefore the sum of their effects measured as above is a minimum for a position of equilibrium.

Let therefore $P_1, P_2, P_3, \ldots P_n$ be n forces acting on a system, and let us conceive (as we always may do) that these forces act from or towards certain points in the lines of their directions, which points we will, after Lagrange, call centres of force, and let $p_1, p_2, p_3, \ldots p_n$ be the distances between these centres and the points of application of the forces; it is plain then that the several forces $P_1, P_2, P_3, \ldots P_n$ remain the same in the deranged as in the undisturbed system, so that the only quantities which vary owing to the displacement are $p_1, p_2, p_3, \ldots p_n$. Let

$$\Pi = \Sigma \, . \, \mathrm{P}p.$$

then in a state of equilibrium H is a minimum, and therefore

$$\delta\Pi = 0$$

$$\therefore \Sigma \, . \, \mathrm{P}\delta p = 0 \qquad (1)$$

that is, the sum of the products of all the forces and the virtual velocities is equal to zero.

IV. Other circumstances of equilibrium.

Are there not however other ways in which the forces may be combined so that $\Sigma \, . \, \mathrm{P}\delta p = 0$, and yet these not be such states of equilibrium as the one we have supposed? The theory of infinitesimals, as applied to the discussion of curves, shows that $\delta\Pi = 0$, when Π is a constant, as well as when Π is a maximum or a minimum—a maximum *or* a minimum I say, because in either case we have such a position of equilibrium as has been supposed in the last section; the absolute sum of the effects of the forces having been taken, without reference to change of sign; for (to adopt the ordinary language of mathematicians) if a body or a system of particles is in equilibrium, whether stable or unstable, an additional force is required to keep it in a position slightly deranged from its position of equilibrium, and therefore the absolute sum of the effects of the forces is, as before stated, greater in the displaced than in the undisturbed system by twice the force which is required to keep the system in its displaced position; but suppose Π to be a constant, the criterion of equilibrium is satisfied, and the body or the system is in equilibrium in other positions of slight displacement; for although by the first law of motion a force is required to change the position in which the body rests, yet as Π is constant, not admitting of increase or decrease owing to any small displacement, we have no force to call into action to be equal to and counterbalance the external force we have applied to move the system, and therefore the system or body would continue changing its position until some other force (as friction, resistance of the air, &c.) acts to neutralize the applied force, and thus to produce rest; whence it is manifest that as soon as such a force does act the system or body is brought to rest and there remains; and as this counteracting force may be brought into action at any instant of the body's motion, it follows that there are an infinite number of positions in which the body or system can rest under the action of the impressed forces. Such a state is what is usually called "Neutral Equilibrium," but what may well be named "a position of stationary action."

V. Method of applying the formula.

The difficulty in applying the equation of virtual velocities to the solution of statical and dynamical problems consists in the calculation of the virtual velocities $\delta p_1, \delta p_2, \ldots \delta p_n$ for the several points of application of the forces. The simplest method, however, is as follows,—to consider two different positions of the system which are consistent with the geometrical relation of the parts, one of equilibrium, the other slightly deranged, and to express the virtual velocities of the points of application of the forces in terms of the coordinates of the points of application and of arbitrary quantities which have been introduced in deranging the system, and as these latter are indeterminate, equating to zero their several coefficients, we shall have a sufficient number of equation to determine the position and conditions of equilibrium.

Let then $P_1, P_2, P_3, \ldots P_n$ be the forces acting on the system, $x_1 y_1 z_1$, $x_2 y_2 z_2, \ldots x_n y_n z_n$ be the coordinates to their points of application, $a_1 b_1 c_1$, $a_2 b_2 c_2, \ldots a_n b_n c_n$, be the coordinates to their centres, defined as in Art. III., and let their direction-cosines be $\cos\alpha_1$, $\cos\beta_1$, $\cos\gamma_1$, $\cos\alpha_2$, $\cos\beta_2$, $\cos\gamma_2, \ldots \cos\alpha_n$, $\cos\beta_n$, $\cos\gamma_n$, and let the distances of their centres from their points of application be $p_1, p_2, p_3, \ldots p_n$, so that we have the following system of equations,

$$\left.\begin{array}{l} p_1^2 = (x_1-a_1)^2 + (y_1-b_1)^2 + (z_1-c_1)^2 \\ p_2^2 = (x_2-a_2)^2 + (y_2-b_2)^2 + (z_2-c_2)^2 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ p_n^2 = (x_n-a_n)^2 + (y_n-b_n) + (z_n-c_n)^2 \end{array}\right\} \quad (2)$$

it is manifest that in any small derangement of the system, such as we have conceived, we may suppose the centres of force to remain fixed, and the points of application to move; but before we proceed further we must make some remarks of great importance.

VI. Analysis of motion into that of translation and that of rotation.

Suppose a body or a system of points of invariable form to change its position in space, and consider two successive positions of the body independently of the forces acting and of the time consumed in passing from one position to the other, it is manifest that there is an infinite number of different ways in which the change may have taken place. Which, then, of all these is the most simple? Which is the best adapted to the methods we have adopted of determining position?

Whatever displacement a system may have undergone, we may conceive the motion to have taken place as follows: 1st, All the points may have moved over equal and parallel lines in the same direction; 2dly, Considering some one point in the body or system to be fixed, and to be a centre round which the body turns, though this point may be continually changing, yet we may give such a revolution to the body or the system, that the several parts of it shall after these two motions hold relatively to space that position which the body has after its real motion; and 3dly, There may be a motion of the particles of the body or system relative to one another, such as to cause a change of the mutual position, influences, and distances on each other, as is the case with gases, elastic fluids, machines in which one part slides on another; this however not being the case with rigid bodies, to which alone it is the object of the present paper to apply the principle of virtual velocities, and as the position of a body in space, with reference to fixed axes and planes, is not affected by this third kind of motion, we shall at present confine ourselves to the first two, into which, though they be of distinct character, all motion, even the most general, may be resolved.

And as the principle of virtual velocities is true for the most general motion conceivable, the velocities are to be estimated in their utmost generality, and therefore the system must receive a motion both of translation and rotation, though not simultaneously but at successive times, as the principle of superposition of small motions, which is equivalent to that of infinitesimals, authorizes; and $\Sigma.P\delta p$ is to be calculated for both these motions; we proceed, therefore, 1st, to give a small motion of translation to the whole system, along each of the three rectangular coordinate axes, whereby all the points of application of the forces will be moved over equal small spaces parallel to the axes; and 2dly, we will turn the body or system successively round the three axes through small angles, the effect of which two motions will be that the points of application of the forces will be moved in the most general way possible, and we shall derive from the results all the conditions of equilibrium.

VII. Application of the principle to motion of translation.

Recurring to formulæ (2) Art. V., and taking the variations, we have

$$\delta p_1 = \frac{x_1 - a_1}{p_1}\delta x_1 + \frac{y_1 - b_1}{p_1}\delta y_1 + \frac{z_1 - c_1}{p^1}\delta z_1$$

and substituting the values of the direction-cosines, we have

$$\delta p_1 = \cos a_1 \delta x_1 + \cos\beta_1 \delta y_1 + \cos\gamma_1 \delta z_1$$

and similarly

$$\left.\begin{array}{l} \delta p_2 = \cos a_2 \delta x_2 + \cos\beta_2 \delta y_2 + \cos\gamma_2 \delta z_2 \\ \quad\ldots\ldots\ldots\ldots\ldots\ldots \\ \delta p_n = \cos a_n \delta x_n + \cos\beta_n \delta y_n + \cos\gamma_n \delta z_n \end{array}\right\} \qquad (3)$$

where δp_1, δp_2 . . . δp_n are the distances relative to the direction of the action of the forces over which the points of application of the forces are moved.

Which may also thus be shown—

Let δs be the absolute displacement of the point $(x_1\, y_1\, z_1)$ owing to the motion of translation, then the direction-cosines of this line δs are $\frac{\delta x}{\delta s}, \frac{\delta y}{\delta s}, \frac{\delta z}{\delta s}$, and if $\phi =$ the angle between the line δs and P's direction,

$$\begin{aligned} \delta p &= \delta s \cos\phi \\ &= \delta s\left\{\frac{\delta x}{\delta s}\cos a + \frac{\delta y}{\delta s}\cos\beta + \frac{\delta z}{\delta s}\cos\gamma\right\} \\ &= \delta x \cos a + \delta y \cos\beta + \delta z \cos\gamma \end{aligned}$$

Applying these values in the equation (1) of virtual velocities, and remembering that all parts of the system move over equal and parallel distances in the same directions, and therefore

$$\left.\begin{array}{l} \delta x_1 = \delta x_2 = \delta x_3 = \ldots\ldots = \delta x_n \\ \delta y_1 = \delta y_2 = \delta y_3 = \ldots\ldots = \delta y_n \\ \delta z_1 = \delta z_2 = \delta z_3 = \ldots\ldots = \delta z_n \end{array}\right\} \qquad (4)$$

the equation $\Sigma . P \delta p = 0$ becomes

$$\Sigma . P \{\cos a \, \delta x + \cos \beta \, \delta y + \cos \gamma \, \delta z\} = 0$$

or, as we may write the equation on account of conditions (4),

$$\Sigma . P \cos a \, \delta x + \Sigma . P \cos \beta \, \delta y + \Sigma . P \cos \gamma \, \delta z = 0 \qquad (5)$$

δx, δy, δz being outside of the sign of summation, and putting

$$\Sigma . P \cos a = X \qquad \Sigma . P \cos \beta = Y \qquad \Sigma . P \cos \gamma = Z$$

$$X \delta x + Y \delta y + Z \delta z = 0 \qquad (6)$$

If then the system or body is entirely free to move in space, we have no other relation given between x, y, z, and their variations, besides (5) or (6), and therefore we have

$$\Sigma . P \cos a = 0 \qquad \Sigma . P \cos \beta = 0 \qquad \Sigma . P \cos \gamma = 0 \qquad (7)$$

that is, the sum of the parts of the forces resolved severally along the three axes of coordinates must separately be equal to zero.

For the cases where the system is not free, but confined to move on a given curve, and for other deductions from (5), reference may once for all be made to the Mécanique Analytique.

If the forces P_1 P_2 . . . P_n act in parallel directions, we have

$$\begin{array}{l} a_1 = a_2 = a_3 = \ldots\ldots = a_n \\ \beta_1 = \beta \ = \beta_3 = \ldots\ldots = \beta_n \\ \gamma_1 = \gamma_2 = \gamma_3 = \ldots\ldots = \gamma_n \end{array}$$

and conditions (7) are reduced to the single one,

$$P_1 + P_2 + P_3 + \ldots\ldots + P_n = \Sigma . P = 0 \qquad (8)$$

VIII. Application of the principle to motion of rotation.

The theory of rotatory motion, as laid down by Poinsot, authorizes us to conclude that every motion or tendency to motion round any axis may be resolved into three motions round three rectangular axes, passing through some point in the axis about which the body turns or has a tendency to turn; and hence it follows, that if we turn a system through small angles round each of three rectangular axes, the resultant of these several motions is equivalent to the most general motion of rotation that the system can receive round any axis passing through the origin of coordinates.

Let $x_1\,y_1\,z_1,\ x_2\,y_2\,z_2,\ \ldots\ x_n\,y_n\,z_n$ be the coordinates to the points of application of the forces $P_1, P_2, P_3, \ldots P_n$; and let

$$\left.\begin{aligned} x_1^2 + y_1^2 + z_1^2 &= r_1^2 \\ x_2^2 + y_2^2 + z_2^2 &= r_2^2 \\ \cdots\cdots & \\ x_n^2 + y_n^2 + z_n^2 &= r_n^2 \end{aligned}\right\} \qquad (9)$$

and let the projections of these several radii vectores be on the plane of

$$\begin{array}{llllll} yz & r_{1x} & r_{2x} & r_{3x} & \ldots & r_{nx} \\ zx & r_{1y} & r_{2y} & r_{3y} & \ldots & r_{ny} \\ xy & r_{1z} & r_{2z} & r_{3z} & \ldots & r_{nz} \end{array}$$

Let us first turn the system through a small angle $\delta\theta$ round the axis of x, θ being the angle between r_x and the axis of y,

$$\begin{aligned} y &= r_x \cos\theta & z &= r_x \sin\theta \\ \delta y &= -\, r_x \sin\theta\delta\theta & \delta z &= r_x \cos\theta\delta\theta \\ \therefore\ \delta y &= -\, z\delta\theta & \delta z &= y\delta\theta \end{aligned}$$

taking the variations of y, z, θ, which are the types of $y_1\,y_2 \ldots y_n$, $z_1\,z_2 \ldots z_n$, $\theta_1\,\theta_2 \ldots \theta_n$.

Similarly let the system be turned through a small angle $\delta\phi$ round the axis of y, ϕ being the angle between r_y and the axis of z, and we have

$$\delta z = -\, x\delta\phi \qquad \delta x = z\delta\phi$$

and let the system be turned through a small angle $\delta\psi$ round the axis of z, ψ being the angle between r and the axis of x, and we have

$$\delta x = -\, y\delta\psi \qquad \delta y = x\delta\psi$$

x, y, z, ϕ, and ψ being, as before, types of the several quantities which correspond to the several forces.

And by the principle of superposition of small motions, being authorized to neglect these quantities which are variations of variations, inasmuch as they become infinitesimals of a higher order, and therefore inappreciable in an expression which involves infinitesimals of a lower order, the total variation of these several points of application of the forces is equal to the sum of the variations due to the several rotations round the several axes of coordinates; hence we have the following typical values of the variations

$$\left.\begin{aligned} \delta x &= z\delta\phi - y\delta\psi \\ \delta y &= x\delta\psi - z\delta\theta \\ \delta z &= y\delta\theta - x\delta\phi \end{aligned}\right\} \qquad (10)$$

and since the expression (3) gives

$$\delta p = \cos a \delta x + \cos \beta \delta y + \cos \gamma \delta z$$

substituting the values (10) we have

$$\delta p = (y \cos \gamma - z \cos \beta)\delta\theta + (z \cos a - x \cos \gamma)\delta\phi + (x \cos \beta - y \cos a)\delta\psi$$

whence substituting again in (1), remembering that $\delta\theta$, $\delta\phi$, $\delta\psi$ are the same for all the forces, and therefore may be written outside the signs of summation, we have finally

$$\Sigma . P(y \cos\gamma - z \cos\beta)\delta\theta + \Sigma . P(z \cos a - x \cos\gamma)\delta\phi + \Sigma . P(x \cos\beta - y \cos a)\delta\psi = 0 \quad (11)$$

and if the system of the body be entirely free to revolve round any of these axes, that is, if we have no other relation between θ, ϕ, and ψ than equation (11), then θ, ϕ, ψ being independent of each other, putting

$$\left.\begin{aligned} \Sigma . P(y \cos \gamma - z \cos \beta) &= L \\ \Sigma . P(z \cos a - x \cos \gamma) &= M \\ \Sigma . P(x \cos \beta - y \cos a) &= N \end{aligned}\right\} \quad (12)$$

we have $L = 0$, $M = 0$, $N = 0$ as the conditions of equilibrium of rotation; that is, since L, M, N are the sum of the moments of the forces tending to turn the system round the three rectangular axes of coordinates then of x, y, z respectively, we conclude that if there is equilibrium the sum of these moments must severally and separately be equal to zero.

Thus then having deduced from the equation of virtual velocities the equations (6) and (11), the six equations of equilibrium which are ordinarily given in treatises on mechanics follow immediately in the manner we have indicated.

Addition to the foregoing Paper.

Suppose the forces which we have been considering in the last paper to act in parallel direction, then we have

$$\left.\begin{aligned} a_1 = a_2 = a_3 = \ldots &= a_n \\ \beta_1 = \beta_2 = \beta_3 = \ldots &= \beta_n \\ \gamma_1 = \gamma = \gamma_3 = \ldots &= \gamma_n \end{aligned}\right\} \quad (1)$$

and the equations (7) are reduced to the single one,

$$P_1 + P_2 + P_3 + \ldots + P_n = 0$$

and equations (12) become

$$\left.\begin{aligned} \cos \gamma \, \Sigma . Py - \cos \beta \, \Sigma . Pz &= 0 \\ \cos a \, \Sigma . Pz - \cos \gamma \, \Sigma . Px &= 0 \\ \cos \beta \, \Sigma . Px - \cos a \, \Sigma . Py &= 0 \end{aligned}\right\} \quad (2)$$

whence we have

$$\frac{\Sigma\,.\,\mathrm{P}x}{\cos\alpha}=\frac{\Sigma\,.\,\mathrm{P}y}{\cos\beta}=\frac{\Sigma\,.\,\mathrm{P}z}{\cos\gamma}=\sqrt{\{(\Sigma\,.\,\mathrm{P}x)^2+(\Sigma\,.\,\mathrm{P}y)^2+(\Sigma\,.\,\mathrm{P}z)^2\}} \qquad (3)$$

Now conceive the parallel forces and their points of application to be so arranged that

$$\Sigma\,.\,\mathrm{P}x=0 \qquad \Sigma\,.\,\mathrm{P}y=0 \qquad \Sigma\,.\,\mathrm{P}z=0 \qquad (4)$$

then α, β, γ become indeterminate, and there is no pressure of rotation, no tendency to make the system turn round any axis, whatever be the direction of the forces relative to the coordinate axes.

Whence it appears that if a system of parallel forces subject to the conditions expressed in equations (1) and (4) act on a body, the system is in equilibrium, whatever be the direction in which the forces act; and this leads to an easy method of determining the centre of parallel forces.

Conceive a number of forces of which P is the type to be acting on a body, xyz being the types of the coordinates of their points of application, and conceive the forces to be so arranged that we have one of them $\overline{\mathrm{P}}$, the coordinates of whose point of application are $\overline{x\,y\,z}$, to be such in intensity and position that the conditions (1) and (4) are fulfilled, viz.

$$\begin{aligned}
\overline{\mathrm{P}} &= \mathrm{P}_1+\mathrm{P}_2+\;.\;.\;.\;.\;+\mathrm{P}_n=\Sigma\,.\,\mathrm{P}\\
\overline{\mathrm{P}x} &= \mathrm{P}_1x_1+\mathrm{P}_2x_2+\;.\;.\;.\;.\;+\mathrm{P}_nx_n=\Sigma\,.\,\mathrm{P}x\\
\overline{\mathrm{P}y} &= \mathrm{P}_1y_1+\mathrm{P}_2y_2+\;.\;.\;.\;.\;+\mathrm{P}_ny_n=\Sigma\,.\,\mathrm{P}y\\
\overline{\mathrm{P}z} &= \mathrm{P}_1z_1+\mathrm{P}_2z_2+\;.\;.\;.\;.\;+\mathrm{P}_nz_n=\Sigma\,.\,\mathrm{P}z
\end{aligned}$$

then we have

$$\bar{x}=\frac{\Sigma\,.\,\mathrm{P}x}{\Sigma\,.\,\mathrm{P}} \qquad \bar{y}=\frac{\Sigma\,.\,\mathrm{P}y}{\Sigma\,.\,\mathrm{P}} \qquad \bar{z}=\frac{\Sigma\,.\,\mathrm{P}z}{\Sigma\,.\,\mathrm{P}}$$

Whence it is easy to determine the coordinates to the centre of gravity of a body, for inasmuch as it is that point at which if the whole weight of the body be collected the moment of rotation about a given axis is the same as the moment of the body constituted as it is, we have a case precisely similar to the one supposed above, and therefore if $\overline{x\,y\,z}$ are the coordinates to the centre of gravity, we have

$$\bar{x}=\frac{\int x\,dm}{\int dm} \qquad \bar{y}=\frac{\int y\,dm}{\int dm} \qquad \bar{z}=\frac{\int z\,dm}{\int dm}$$

On infinitesimal Analysis.

The calculus of infinitesimals appears to be a branch of analysis to which much attention has not been paid by English writers. The method of limits has probably been supposed more tangible and more satisfactory to minds previously unacquainted with considerations of this nature; and indeed it is not without difficulty that the mind first forms a conception of a system of quantities differing infinitely from one another in magnitude, and yet all distinct from an absolute zero, and learns that many mathematical expressions are formed only approximately by means of neglecting terms of certain orders of infinitesimals. Now although this is the case, there still exists a difficulty of a not entirely different nature in the doctrine of limits, so that the latter method is perhaps not so eminently superior to that of infinitesimals as to demand the complete exclusion of the latter. In fact its frequent employment in applied mathematics, as in the theory of curves of double curvature, of small oscillations, including the undulatory theory of light, and other cases, appears to call for some explanation of its fundamental principles. Moreover, the two methods obviously leading to the same results must in fact come to the same thing; and it may not be inappropriate here to mention the point at which the coincidence takes place. The limit of any continually increasing or decreasing quantity or ratio is that quantity or ratio to which it always tends, but to which although it may approach nearer than by any finite quantity or ratio, it never becomes actually equal. The difficulty then consists in passing from any given value of the quantity or ratio under consideration to the limit, for in doing this (however near we may approximate to the limit) a saltus must finally be made; because the essence of a limit consists in this, that by no continuous approximation is it possible ever to arrive at it. Now it is obvious that nothing short of the limit itself will give accurate expressions in the result, and, consequently, it is necessary either to reach the absolute limit or to remain satisfied with approximations. The former course is pointed out by the method of limits, the latter by that of infinitesimals. In order to compare the two methods it will be well to consider a simple example. In both cases a curve is considered to be approximately a polygon, the degree of approximation increasing with the number of the sides ; and in the method of limits it is said that the two figures coincide accurately when the number of sides becomes infinite, or, in other words, that in the limit the magnitudes of the sides absolutely vanish ; while, in the infinitesimal theory, it is supposed that the magnitudes of the sides always have an existence, and that they themselves are indefinitely divisible. Now, if it be required to consider the deflexion of the curve from the tangent, that is, the angle between any two consecutive tangents, it will, in the former theory, be necessary generally to consider the second tangent as drawn through some point at a small distance from the first point, and afterwards again to have recourse to the limit, by making the distance between the two points actually vanish. Hence, geometrical representations can be formed of an infinitesimal no less than of a finite quantity, because they differ from one another only in magnitude ; but since in the limit all magnitude is lost, no complete geometrical representation can be formed : and, in fact, (returning to the example noticed above,) as long as any appreciable angle exists, in general between the two tangents (the two points of contact actually coinciding) the curve becomes a polygon, and we find ourselves employing the infinitesimal method. It therefore appears that the method of limits is only a particular form of that of infinitesimals, and that, from the geometrical indeterminateness arising from the notion of the limit, the infinitesimal quantities are not actually destroyed or eliminated, but only lost sight of.

Perhaps no better authority for the definition of infinitesimals and their use can be quoted than Carnot, who makes the following remarks : " J'appelle *quantité infiniment petite,* toute quantité qui est considerée " comme continuellement décroissante, tellement qu'elle puisse être " rendue aussi petite qu'on le veut, sans qu'on soit obligé pour cela, de " faire varier celles dont on cherche la relation. Lorsqu'on veut " trouver la relation de certaines quantités proposées, les unes con- " stantes, les autres variables, on considère le système général comme " parvenu à un état déterminé que l'on regarde comme fixe : puis on

" compare ce système fixe avec d'autres états du même système, " lesquels sont considérés comme se rapprochant continuellement du " premier, jusqu'à en différer aussi peu qu'on le veut. Ces autres états " du système ne sont donc à proprement parler eux-mêmes, que des " systêmes auxiliaires, que l'on fait intervenir pour faciliter la com- " paraison entre les parties du premier. Les différences des quantités " qui se correspondent entre tous ces systèmes peuvent donc être sup- " posées aussi petites qu'on le veut, sans rien changer aux quantités " qui composent le premier, et qui sont celles dont on cherche la " relation. Ces différences sont donc de la nature des quantités que " nous appelons *infiniment petites :* puisqu'elles sont considérées comme " continuellement décroissantes, et comme pouvant devenir aussi petites " qu'on le veut, sans que pour cela, on soit obligé de rien changer à " la valeur de celles dont on cherche la relation." " *L'analyse infinité-* " *simale* n'est autre chose que l'art d'employer auxiliairement les " quantités infinitésimales, pour découvrir les relations qui existent " entre des quantités proposées." * In addition to which let it be observed that the ratio of an infinitesimal of the $(n+1)$th order to one of the nth is an infinitesimal of the first order, and the ratio of any two infinitesimals of the same order is finite.

A quantity or function is said to vary continuously from one value to another when it is capable of receiving any value between the given ones as its limits. In this it is to be observed, that any values (which lie between the limits) chosen arbitrarily and in any order of succession may be given to the quantity or function without violating the laws to which it is subject; and consequently the difference between any two consecutive values may itself have any value not greater than the difference between the two limits. This difference may, however, obviously be made as small as is desired by taking the two consecutive values sufficiently near to one another; and by taking the same value twice in succession, it may be made absolutely zero. Hence a series of these differences, considered as a system of quantities, are as arbitrary as the values themselves, and their differences may consequently have any value not greater than the greatest of the first differences. These new

* Carnot, Reflexions sur la Metaphysique du Calcul Infinitésimal, Chap. I.

differences (or, as they will be called, *second* differences) may also obviously be made as small as the difference between the first differences whose values approach nearest to one another in value; and as small as is desired, and even zero, by a proper choice of the first differences, if the latter are independent, i. e. subject to no conditions. This process, being evidently subject to no other limitations than those above given, may be continued as long as is desired; and a difference of any order, of the nth for example, is consequently as arbitrary in magnitude as the first assumed values, provided always that no conditions exist which can affect the differences of any order inferior to the nth.

Let these differences be represented in the following manner; suppose that u is any quantity or function, and u u' two values of u, then their difference

$$u' - u = \mathrm{d}u;$$

similarly if $\mathrm{d}u$, $(\mathrm{d}u)'$ be two values of $\mathrm{d}u$, then

$$(\mathrm{d}u)' - \mathrm{d}u = \mathrm{dd}u;$$

and similarly the successive differences of u may be thus represented

$$\mathrm{d}u, \quad \mathrm{dd}u, \quad \mathrm{ddd}u, \; \ldots\ldots$$

Now it is clear that the superior limit of the magnitude of any difference of the nth order depends upon the magnitude of the greatest difference of the $(n-1)$th order, this again on that of the $(n-2)$th, and so on until that of the $(n-n)$th, i. e. until the given values themselves; but the inferior limit of the magnitude of any difference of the nth order depends upon the distribution of the magnitudes of the differences of the $(n-1)$th, these again on those of the $(n-2)$th, and so on until the $(n-n)$th, i. e. finally upon the distribution of the original values chosen between the given limits. Consequently when the given limits differ infinitely from one another the differences of any given order, as for instance the nth, may be made infinitely great, finite, or infinitely small; while in every case they may be made infinitely small. These remarks will be sufficient to explain the general theory of the various orders of infinitesimals.

It was seen above that by choosing the primary arbitrary values of the given quantity or function sufficiently near one another, the first differences may be made as small as are required; in other words, they may be made infinitely small in comparison with the values themselves; in this case they will be called differentials. Again, by repeating the process the second differentials may, in accordance with the remarks made above, be made infinitely small in comparison with the first differentials; and so on without limit.

It must however be remembered that these differentials, although infinitesimals of any order, however great, are still actually different from zero; because it is only in the case where two consecutive values of the $(n-1)$th differences have been taken actually equal to one another that the nth differences vanish; which is a case distinct from the one indicated above. These remarks are evidently applicable whatever be the order of infinitude of the primary quantity or difference under consideration. Hence, of any quantity, whether infinite, finite, or infinitesimal, there may be an infinite number of differentials, each differing from one another by any required order of infinitude. Now these differences or differentials of the orders 1, 2, 3, ... exist, as has been seen, when the function under consideration varies continuously from one given value to another between its given limits. Similarly those of the orders 2, 3, 4, ... exist when the first differential varies continuously between its proper limits; and so on for others. But on the other hand, these differentials evidently may exist also in the case when the function varies no longer continuously but per saltus, never receiving any other values than those contained in a given system. For the differences of these fixed values will be the differences of the first order, and the differences of these first differences (if the first differences be unequal, as must be supposed in the general case,) those of the second order. The differences will however no longer be arbitrary, as in the case of continuous variations, but will be determinate as soon as the saltus of the primary functions are given. But in order to make a differential of any given order, as the nth for instance, vanish, it will be obviously necessary to perform upon the differentials of the $(n-1)$th order an operation which is only a particular case of the general one above considered, and differing, not in nature but in *intensity* (if it may be so termed), from that which was performed upon those of the $(n-2)$th. Since however this would introduce much complexity, especially as in the various problems which present themselves it would be convenient to employ a larger or smaller number of orders of differentials according to circumstances (which would destroy that uniformity of method which it is always desirable to combine with an unity of principle), the method usually adopted, and the one which appears most desirable for general use, is the following; that the differentials of any given orders, as of $m, n, p, \ldots$ should coincide in magnitude with infinitesimals of the same orders respectively; hence, the precisely same operation which converts a differential of the nth order into one of the $(n+1)$th will convert one of the $(n+1)$th into one of the $(n+2)$th; and so generally r operations of the same nature will convert a differential of the nth order into one of the $(n+r)$th.*

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* An example of this is the theorem of Taylor for the expansion of $f(x+dx)$ in a series whose well known form is,

$$f(x+dx)=f(x)+f(x)\frac{dx}{1}+f''(x)\frac{dx^2}{1\,.\,2}+\ldots\ldots$$

the number of terms being infinite. The usual expression for the remainder after n terms is

$$\frac{dx^n}{1\,.\,2\,.\,3\,.\,.\,.\,n}f^{(n)}(x+\theta dx)$$

where θ is some quantity between 0 and 1. If however

$$'f^{(n)}(x)$$

represent that the $(n+1)$th differential of $f(x)$ is to be made to vanish, the series in question might be also written,

$$f(x+dx)=f(x)+f'(x)\frac{dx}{1}+f''(x)\frac{dx^2}{1\,.\,2}+\ldots+\frac{dx^n}{1\,.\,2\,.\,3\,.\,.\,.\,n}\,'f^{(n)}(x)$$

which however is not so convenient as the former, since a less clear idea of the remainder after n terms is given by the new operation $'f$ than by the quantity θ.

Another instance of various symbols of operation is one proposed by M. E. Lamarle in Liouville's Journal de Mathématiques, Juillet 1846; it is as follows;

" Lorsqu'on écrit

$$f(x+h)-f(x)=hf'(x+\theta h), \tag{1}$$

" on se borne, en général, à faire observer que θ désigne une quantité comprise entre 0 et 1.
" On sait cependant que les premières notions de l'analyse algébrique permettent de fixer
" d'une manière extrêmement simple et tout à fait précise le sens de l'équation (1).
" Ajoutons qu'en l'écrivant sous la forme que nous venons d'indiquer, on est forcé d'en
" restreindre l'application aux cas où la fonction et sa dérivée demeurent continues dans
" l'intervalle que l'on considère.

" Cette remarque suffira sans doute pour justifier la préférence que nous accordons à
" la formule suivante:

$$f(x+h)-f(x)=h\mathfrak{M}_x^{x+h}f'(x), \tag{2}$$

" où la caractérisque $\mathfrak{M}$ se trouve définie par l'équation de condition

$$\mathfrak{M}^{x+h}f'(x)=\lim\left[\frac{f'\left(x+\frac{h}{n}\right)+f'\left(x+2\frac{h}{n}\right)+f'\left(x+3\frac{h}{n}\right)+\ldots+f'(x+h)}{n=\infty}\right]$$

" et qui peut s'énoncer en ces termes: 'Dans tout intervalle *où la fonction demeure con-*
" *tinue,* son accroissement a pour mesure l'accroissement de la variable, multiplié par la
" *valeur moyenne* de la fonction dérivée.' "

There are however certain particular cases in which this process, viz. of making the differentials of certain orders vanish, is used with great advantage; as for instance, if there are $(n-1)$ equations involving n variables and their differentials of various orders, any one of the differentials of any of the variables may always be determined in terms of the remaining differentials of the $(n-1)$ variables, the differentials of the nth variable, and the variables themselves. Consequently one of the differentials of the nth variable is arbitrary, i. e. subject to no restriction; it may therefore receive any value at pleasure; the value usually assumed is zero; and the differential equated to zero is usually the second. The variable subjected to this arbitrary condition is called the independent variable; and it is easily seen that any one of the n variables may be so treated. It is also observable that unless second differentials are involved there is no clue by which we can discover the independent variable. It may further be observed that the second differential has in preference to all others been subjected to the condition above stated, because it is of the lowest order which we are at liberty so to treat. This will easily be seen when we consider that from the manner in which these differentials have been formed it appears that the giving the differentials of any order (the mth for instance) any arbitrary value (such as zero) is a condition which affects not only the differentials of the mth and all higher orders, but also those of the $(m-1)$th order, inasmuch as the values of the differentials of the latter order to be chosen for the formation of those of the mth are no longer arbitrary, but restricted to equidistant values. Hence the first differentials could not be made to vanish without subjecting the values of the variables themselves to limitations which the conditions of the problem would not justify. It is also clear that if the second differentials vanish, the third, fourth, ... and all others do so also. This is the whole theory of the independent variable.

In actual practice it will of course be impossible to retain an infinite number of orders of differentials, as we should in that case have an infinite number of quantities to contemplate and take account of at the same time, and our faculties are such that this is beyond their power; hence we are obliged to have recourse to a method of approximation, and the one naturally adopted is to neglect those small quantities which do not affect the truth of the results. This may at first sight appear an unsatisfactory method of operation, and one which overturns all the preconceived notions of the absolute truth and correctness of mathematical results ; but it is nevertheless the method actually adopted (or at least equivalent to it), although often brought to notice under different aspects. But although this is the case, it is still equally true that the error committed by this method can never become appreciable, for the neglect of infinitesimals of any order, as those of the nth for instance, can for each term so discarded produce no errors of an order lower than the nth; but no sum of a finite number of infinitesimals of the nth order can produce a quantity of the $(n-1)$th; similarly no sum of a finite number of infinitesimals of the $(n+1)$th order can produce a quantity of the nth; and so on throughout all orders. Suppose now, for example, that above the $(n+r)$th order none but finite sums exist, or in other words that by neglecting differentials of this order no quantities of lower orders are discarded, (n and r may, of course, have any value from $-\infty$ to $+\infty$). Now the only case where an error of an order lower than the $(n+r)$th can be committed by neglecting terms of this and higher orders is that in which there are an infinite number of terms of this order; but as in no expression which can occur in practice is it possible to write down these at full length, the terms in question must themselves (if they exist at all) appear under some other form (such as a sum or product, for example), which in fact will present them as a quantity of the $(n+r)$th or some lower order; but as terms of the latter order have by hypothesis been retained, no error of an order lower than the $(n+r)$th will have been committed; and as terms of this order have been considered as inappreciable, the possible errors will themselves also be inappreciable. On the other hand, if after having retained no quantities of orders higher than the $(n+r)$th it be thought desirable to introduce those of the $(n+r+1)$th, a similar train of reasoning will be applicable, and no theoretical difficulty will arise. It has now been shown that the usual method of differentials is a convenient one, that if it be adopted terms of certain orders must be neglected, but that in neglecting these terms no error of any order lower than the lowest neglected order can be committed; the extent of the accuracy of the investigations is consequently always determinate, and the method when carefully pursued is free from danger.

When the orders of differentials follow those of the infinitesimals, as indicated above, they will be denoted by the symbol d, so that the successive differentials of any quantity or function u will be thus denoted,

$$du,\ ddu,\ dddu, \dots\dots$$

or as they are more usually written,

$$du,\ d^2u,\ d^3u, \dots\dots \tag{1}$$

Since we are best acquainted with quantities which are called finite, and less so with those which are indefinitely greater or indefinitely less than these, it has been thought advantageous to commence the notation of the powers of the symbol d from this as their zero point. Now since the powers of any symbol (as in the case of d) may be increased in a negative as well as a positive direction, without limit, it will be necessary to determine the meaning of quantities when affected with the symbol of operation d raised to any power.

It must here be recollected that we have not proved, nor do we assume, that the law of the operation indicated by the symbol d follows in general the index or any other law; but since it is known à priori that any mathematical symbol (be it one of quantity, quality, or anything else) is liable to be subjected to any algebraical operations, the same must be the case with the symbol now under discussion, namely d; the question therefore now is, what operation does the symbol d^n represent, (1) when n is a positive integer, (2) when it is a negative integer, (3) when it is fractional. Now the first case is easily determined, for from the manner in which the differentials have been formed, i. e. from their very nature, we see that the symbol d does follow the index law when the index is a positive integer; and this also because we have not given the term differential or differential coefficient any meaning which will require to be afterwards further elucidated (as, for instance, differential coefficient has been defined to be the coefficient of the first power of h in the expansion of $f(x+h)$ in a series of ascending powers of h), but have shown the actual connexion with the operations of differentiation with the fundamental operations of all algebraical processes. For this purpose it is easily seen, that in order to pass from the differential of any quantity of the nth order to that of the same quantity of the $(n+1)$th order, it is sufficient merely to write dd^nu, i.e. $d^{n+1}u$, for d^nu, i.e. multiply the differential of the nth order by the symbol d. Hence, in the method now adopted, the raising of the index of the symbol d by unity turns an infinitesimal of the nth into one of the $(n+1)$th. Similarly, in order to pass from the differential of any quantity of the nth order to one of the $(n-1)$th, it is sufficient merely to divide the differential of the nth order by the symbol d. Hence, in the method now adopted, the decreasing of the index of the symbol d by unity turns an infinitesimal of the nth order into one of the $(n-1)$th. In order however to explain more fully the nature of the operation indicated by the symbol d^{-1} or $\frac{1}{d}$, it will be necessary to revert to the original formation of the quantity du. It was above seen that the various values which can be given to du are formed by taking the differences of the arbitrary values of the given function which lie between those limits within which the function remains continuous. This being the operation indicated by d, it is now required to determine the nature of the inverse operation indicated by d^{-1}. But since by the operation d the quantity under consideration, u, was divided into a number of indefinitely small portions, whose general type was du, and the sum of which was the difference of the values assumed by u at the given limits, the required operation will be to deduce from any given differential, as du_i for instance, the quantity u_i; this will obviously be effected, if to the inferior limit of u (u_o) there be added the sum of so many of the differentials whose form is du as will be equal to the difference between u_o and u_i; or, if more convenient, by subtracting from the superior limit u_n the sum of so many of the differentials as will be equal to the difference between u_n and u_i. This result may also be thus analytically represented:

Suppose u to be any function of x, so that

$$u = f(x)$$

also let

$$f'(x) = f(x+dx) - f(x) = df(x)$$

But

$$\begin{aligned} f'(x) &= f(x+dx) - f(x) \\ f'(x+dx) &= f(x+2dx) - f(x+dx) \\ f'(x+2dx) &= f(x+3dx) - f(x+2dx) \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ f'(x+\overline{i-1}dx) &= f(x+idx) - f(x+\overline{i-1}dx) \end{aligned}$$

Consequently

$$f'(x) + f'(x+dx) + \ldots + f'(x+\overline{i-1}dx) = f(x+idx) - f(x)$$

hence

$$f(x+idx) = f(x) + f'(x) + f'(x+dx) + \ldots + f'(x+\overline{i-1}dx)$$

Now if from the superior limit of $f(x)$ ($f(x_n)$ suppose) there be subtracted the inferior limit ($f(x_o)$ suppose) the result viz. $f(x_n) - f(x_o)$ is called the definite integral of $f'(x)$; suppose moreover that

$$x_o + idx = x_n$$

then

$$f(x_n) - f(x_o) = f'(x_o) + f'(x_o + dx) + \ldots + f'(x_o + \overline{n-1}dx_o) \qquad (2)$$

that is, the definite integral of any function is equal to the sum of the values assumed by the differentials of that function between the given limits, inclusive of the inferior and exclusive of the superior limit. Such then is the nature of the operation indicated by the symbol d^{-1}; and it is also clear that the same may be repeated any number of times without limitation; hence it will not be necessary to consider further the nature of the operations indicated by the symbols whose general type is d^{-m}, where m is any integer. A similar train of reasoning would show that the symbol d may be raised to any positive or negative integral power, and any root of such power be taken, and that the result will always have an intelligible meaning; thus, for instance, if d be raised to the pth power the result will be d^p, and if the qth root of this result be taken the final result will be $d^{\frac{p}{q}}$, and if

$$\frac{p}{q} = r$$

we should have

$$d^{\frac{p}{q}} = d^r$$

that is, the result will be the same as if we had arrived at d^r by a direct process.

The meaning of the operations indicated by the symbols d^n d^{-n}, where n is any integer, having been determined, the following question also suggests itself; what interpretation is to be given to a symbol of the form $d^{\frac{m}{n}}$? In the first place it may be remarked that a quantity affected with the symbol d raised to any power (the nth for instance) is of any order of infinitesimals one degree higher than the same affected with the symbol raised to the $(n-1)$th power. Now this it must be remembered was a particular method which it was thought desirable to employ. Suppose, however, that the symbol d denote the operation of differentiation when there is no limitation respecting the connexion of the orders of differentials and infinitesimals, then, as before, the results of successive operations of this nature on any function u will give the series

$$\mathrm{d}u,\ \mathrm{dd}u,\ \mathrm{ddd}u,\ \ldots\ldots$$

or as they may also be written

$$\mathrm{d}u,\ \mathrm{d}^2u,\ \mathrm{d}^3u,\ \ldots\ldots \qquad (3)$$

in which any conceivable relation may exist between the symbols d and d; but among all of these the one with which we are now concerned is that in which n operations of the nature d are equivalent to m operations of the nature d, i. e.

$$\mathrm{d}^n u = d^m u$$

or as it may be written symbolically

$$\mathrm{d}^n = d^m, \text{ or } \mathrm{d} = d^{\frac{m}{n}}, \text{ or } \mathrm{d}^{\frac{n}{m}} = d \qquad (4)$$

Now it is observable from equation (4) that by performing the operation d n times any given quantity u is changed from a finite quantity to an infinitesimal of the mth order, or that by performing the operation d once u is changed from a finite quantity to an infinitesimal of the $\frac{m}{n}$ order. This is what was required. Now, although we have arrived directly at infinitesimals of fractional order, yet it must not on that account be expected that instances of such quantities can be given; for on account of our inability to determine the exact limits of finite and infinite (whether infinitely large or infinitely small) quantities, we cannot determine whether any given quantity lies at $\frac{1}{n}$th or $\frac{1}{m}$th part of the distance from one to the other. But the same reasoning which shows the possibility (möglichkeit) of the various orders of infinitesimals involves also the necessity (nothwendigkeit) of all orders, and consequently of fractional as well as integral. Thus analysis, being the representation of real laws, gives as some of its results these *partial* infinitesimals, as they may be called; which, since they lie between two given orders of infinitesimals, must have a real existence, although there is an apparent indeterminateness about them. But this indeterminateness is itself a result of the absolute magnitudes of the various orders of infinitesimals which have been chosen as near together as was possible without destroying the general distinctness of the various orders. From the actual existence of these partial infinitesimals it follows that they may be used in any analytical expression with no less propriety than the total infinitesimals; and although the former are not in such general use as the latter, yet there are some cases in which they may be employed with great advantage.

[The following is an example; in vanishing fractions (i. e. when for a given value of x, as

$$x = a$$

the fraction

$$\frac{\mathrm{f}(x)}{f(x)}$$

takes the form $\frac{0}{0}$) certain cases present themselves in which, however many times the numerator and denominator may be differentiated, the value of the fraction still appears indeterminate, so that

$$\frac{\frac{d\mathrm{f}(x)}{dx}}{\frac{df(x)}{dx}} = \frac{0}{0}, \quad \frac{\frac{d^2\mathrm{f}(x)}{dx^2}}{\frac{d^2f(x)}{dx^2}} = \frac{0}{0}, \ldots\ldots$$

The readiest way to evaluate these is to find the general forms of

$$\frac{d^n\mathrm{f}(x)}{dx^n} \text{ and } \frac{d^nf(x)}{dx^n}$$

and then to give n some fractional value, the result will then be found determinate.]

From the manner in which differentials and integrals have been formed, viz. by subtraction and addition, it is obvious that the order in which these are performed when any function is affected with symbols of both operations, is indifferent, so that

$$d^m \textstyle\int^n u = \int^n d^m u \qquad (5)$$

Hitherto functions of only one variable have been considered; but as there are some points in which functions of several variables differ from those of one variable, it will be worth while briefly to notice them. Consider then the function of any number of variables

$$f(x, y, z, \ldots) = 0$$

the total differential of this is obviously

$$f(x + dx,\ y + dy,\ z + dz,\ \ldots) - f(x, y, z, \ldots)$$

But as in certain cases it may happen that only one of the quantities $x, y, z, \ldots$ may be considered variable while the rest remain constant, the partial differentials so formed will be represented by the symbols

$$d_x,\ d_y,\ d_z, \ldots\ldots$$

the suffix denoting the quantity which is considered variable.

Now,

$$d_x f(x, y, z, \ldots) = f(x+dx, y, z, \ldots) - f(x, y, z, \ldots)$$

also

$$d_y d_x f(x, y, z, \ldots) = f(x+dx,\ y+dy,\ z, \ldots) - f(x,\ y+dy,\ z, \ldots) \\ - f(x+dx, y, z, \ldots) + f(x, y, z, \ldots)$$

Again,

$$d_y f(x, y, z, \ldots) = f(x, y+dy, z, \ldots) - f(x, y, z, \ldots)$$

$$d_x d_y f(x, y, z, \ldots) = f(x+dx, y+dy, z, \ldots) - f(x+dx, y, z, \ldots) \\ - f(x, y+dy, z, \ldots) + f(x, y, z, \ldots)$$

hence

$$d_y d_x f(x, y, z, \ldots) = d_x d_y f(x, y, z, \ldots)$$

Similarly it may be proved that

$$d_x d_y d_z \ldots f(x, y, z, \ldots) = d_y d_z d_x f(x, y, z, \ldots) = \ldots \tag{6}$$

or symbolically

$$d_x d_y d_z \ldots = \mathrm{P}[d_x,\ d_y,\ d_z, \ldots] \tag{7}$$

where P represents the continued product of all the symbols taken in any order whatever. Again, the total differential of f may be thus written

$$\left.\begin{array}{ll} f(x+dx,\ y+dy,\ z+dz \ldots) - f(x, y, z, \ldots) & \\ = f(x+dx,\ y+dy,\ z+dz \ldots) & - f(x,\ y+dy,\ z+dz \ldots) \\ + f(x,\ y+dy,\ z+dz \ldots) & - f(x, y, z+dz \ldots) \\ + f(x, y, z+dz \ldots) & - f(x, y, z, \ldots) \\ + \ldots\ldots\ldots\ldots & \ldots\ldots\ldots \\ & - f(x, y, z, \ldots) \end{array}\right\} \tag{8}$$

But

$$f(x+dx,\ y+dy,\ z+dz \ldots) - f(x,\ y+dy,\ z+dz \ldots) \\ = d_x[f(x,\ y+dy,\ z+dz \ldots)]$$

consequently

$$f(x+dx,\ y+dy,\ z+dz \ldots) - f(x,\ y+dy,\ z+dx \ldots) \\ - f(x+dx,\ y,\ z+dz \ldots) + f(x, y, z+dz \ldots) \\ = d_y[f(x+dx, y, z+dz \ldots)] - d_y[f(x, y, z+dz \ldots)] \\ = d_x d_y f(x, y, z+dz \ldots)$$

which is an infinitesimal of the second order; consequently repeating the process for the other variables, it would be at length found that the expression

$$f(x + dx,\ y + dy,\ z + dz\ .\ .\ .) - f(x,\ y + dy,\ z + dz\ .\ .\ .)$$
$$= d_x[f(x,\ y + dy,\ z + dz\ .\ .\ .)]$$

differs from

$$f(x + dx, y, z\ .\ .\ .) - f(x, y, z\ .\ .\ .) = d_x[f(x, y, z\ .\ .\ .)]$$

only by infinitesimals of orders higher than the first, and similar results would be all deducible for the other terms of (8); consequently, as far as infinitesimals of the first order

$$\begin{aligned} f(x + dx, y + dy, z + dz\ .\ .\ .) &- f(x, y, z\ .\ .\ .) \\ = f(x + dx, y, z\ .\ .\ .) &- f(x, y, z\ .\ .\ .) \\ + f(x, y + dy, z\ .\ .\ .) &- f(x, y, z\ .\ .\ .) \\ + \ .\ .\ .\ .\ .\ .\ .\ .\ . \end{aligned}$$

or writing

$$u = f(x, y, z\ .\ .\ .)$$
$$du = d_x u + d_y u + d_z u + .\ .\ .$$

or multiplying and dividing the several terms by dx, dy, dz, . . . respectively, there results the usual formula

$$\mathrm{D}u = \frac{du}{dx}dx + \frac{du}{dy}dy + \frac{du}{dz}dz + .\ .\ .\ .\ .\ .\ .$$

where D represents the total differential and $\frac{du}{dx}$ . . . the ratios of the partial differentials of u (considering as variable only those quantities whose differentials appear in their respective denominators), and the differentials of x, y, z . . ., or as they are usually called the partial differential coefficients of u with respect to x, y, z . . . .

Examples of the Application of the Infinitesimal Calculus.

The following are some miscellaneous examples of the application of the above methods to actual practice.

I.

To find dy in the following cases, calculating the expressions as far as infinitesimals of the first order only.

$y = a + x,$

$$\therefore\ dy = (a+x+dx) - (a+x) = dx.$$

$y = a - x,$

$$\therefore\ dy = (a-x-dx) - (a-x) = -dx.$$

$y = ax,$

$$\therefore\ dy = a(x+dx) - ax = adx.$$

$y = \frac{a}{x},$

$$\therefore\ dy = \frac{a}{x+dx} - \frac{a}{x} = -\frac{adx}{x(x+dx)}$$

$$= -\frac{adx}{x^2}(1 - \frac{dx}{x} + \ .\ .\ .) = -\frac{adx}{x^2}.$$

$y = x^a,$

$$\therefore\ dy = (x+dx)^a - x^a$$

$$= ax^{a-1}dx + \frac{a(a-1)}{1.2}x^{a-2}dx^2 + \ .\ .\ . = ax^{a-1}.$$

$y = a^x,$

$$\therefore\ dy = a^{x+dx} - a^x = a^x(a^{dx}-1)$$

$$= a^x(\log_e adx + \frac{(\log_e a . dx)^2}{1.2} + \ .\ .\ .) = \log_e a . a^x dx.$$

$y = e^x,$

$$\therefore\ dy = e^{x+dx} - e^x = e^x(e^{dx} - 1)$$

$$= e^x(dx + \frac{dx^2}{1.2} + \ .\ .\ .) = e^x dx.$$

$y = \sin x,$

$$\begin{aligned}
\therefore\ dy &= \sin(x + dx) - \sin x \\
&= \sin x\,(\cos dx - 1) + \cos x\ \sin dx \\
&= -\tfrac{1}{2}\sin x\ \sin^2 dx + \ldots + \cos x\ \sin dx \\
&= \cos x\ \sin dx = \cos x dx,
\end{aligned}$$

since $\cos dx = \sqrt{1 - \sin^2 dx} = 1 - \frac{1}{2}\sin^2 dx + \ldots,$

and $\sin dx = dx$, approximately.

$y = \cos x,$

$$\begin{aligned}
\therefore\ dy &= \cos(x + dx) - \cos x \\
&= \cos x\,(\cos dx - 1) - \sin x\ \sin dx \\
&= -\sin x\ \sin dx = -\sin x dx.
\end{aligned}$$

$y = \tan x,$

$$\begin{aligned}
\therefore\ dy &= \tan(x + dx) - \tan x \\
&= \frac{\tan x + \tan dx}{1 - \tan x \tan dx} - \tan x \\
&= \tan dx \frac{1 + \tan^2 x}{1 - \tan x\ \tan dx} = (1 + \tan^2 x)\, dx.
\end{aligned}$$

$y = \frac{x}{z},$

$$\therefore\ dy = \frac{x + dx}{z + dz} - \frac{x}{z} = \frac{z dx - x dz}{z(z + dz)} = \frac{z dx - x dz}{z^2}.$$

II.—TAYLOR'S THEOREM.

Let

$$u = \mathrm{F}(x, y, z, \ldots .)$$

be any function of the variables $x\, y\, z\ \ldots .$; and let

$$x, y, z, \ldots .$$

receive increments

$$dx, dy, dz, \ldots .$$

respectively, and let

$$u' = \mathrm{F}(x + dx, y + dy, z + dz\ \ldots .)$$

If then we consider the increments

$$dx, dy, dz, \ldots .$$

as infinitesimals of the first order, and

$\mathrm{D}u$

a corresponding quantity relative to the function u, calculated as far as the first order only, we shall have as a first approximation

$$u' - u = \mathrm{D}u \tag{1}$$

or

$$u' = u + \mathrm{D}u \tag{A}$$

But since the function u' may involve the products of the differentials $dx\, dy\, dz$ . . . . ., the quantity $\mathrm{D}u$ will not represent accurately the difference $u' - u$, but will require certain terms of higher orders to be added to it. Hence as a second approximation we shall have, in a similar manner

$$u' - u - \mathrm{D}u = \mathrm{D}(\mathrm{D}u) = \mathrm{D}^2u$$

or

$$u' = u + \mathrm{D}u + \mathrm{D}^2u \tag{2}$$

Now neither of the values of u' given by (1) or (2) is a correct one, and that given by (2) may be either too great or too small (according to the signs of the succeeding terms); hence we must take as the most probable approximation the average of the values given by the several approximations: Hence adding (1) and (2) and dividing by 2 (the number of approximations), we have for the next value of u'

$$u' = u + \frac{\mathrm{D}u}{1} + \frac{\mathrm{D}^2u}{1.2} \tag{B}$$

Treating these in a similar way we shall have next

$$u' - u - \frac{\mathrm{D}u}{1} - \frac{\mathrm{D}^2u}{1.2} = \mathrm{D}\left(\frac{\mathrm{D}^2u}{1.2}\right) = \frac{\mathrm{D}^3u}{1.2}$$

or

$$u' = u + \frac{\mathrm{D}u}{1} + \frac{\mathrm{D}^2u}{1.2} + \frac{\mathrm{D}^3u}{1.2} \tag{3}$$

and, as before, adding together (1) (2) (3), and dividing by 3, we find

$$u' = u + \frac{\mathrm{D}u}{1} + \frac{\mathrm{D}^2u}{1.2} + \frac{\mathrm{D}^3u}{1.2.3} \tag{C}$$

Similarly the next value would be

$$u' = u + \frac{\mathrm{D}u}{1} + \frac{\mathrm{D}^2u}{1.2} + \frac{\mathrm{D}^3u}{1.2.3} + \frac{\mathrm{D}^4u}{1.2.3.4} \tag{D}$$

until we should have as the nth approximation

$$u' = u + \frac{\mathrm{D}u}{1} + \frac{\mathrm{D}^2u}{1.2} + \frac{\mathrm{D}^3u}{1.2.3} + \ldots + \frac{\mathrm{D}^nu}{1.2.3\ldots n} \tag{N}$$

Moreover, carrying on the approximations in a similar manner *ad infinitum*, the remaining terms of the series giving the accurate value of u' in terms of u and its differentials will be

$$\frac{1}{1.2.3\ \ldots\ldots\ n(n+1)} D^{n+1}\left\{u + \frac{Du}{n+2} + \frac{D^2u}{(n+2)(n+3)} + \ldots\ldots\right\}$$

which quantity is evidently

$$> \frac{1}{1.2.3\ \ldots\ldots\ (n+1)} D^{n+1}u$$

and

$$< \frac{1}{1.2.3\ \ldots\ldots\ (n+1)} D^{n+1}u'$$

i. e.

$$> \frac{1}{1.2.3\ \ldots\ldots\ (n+1)} D^{n+1}F(x, y, z\ \ldots)$$

and

$$< \frac{1}{1.2.3\ \ldots\ldots\ (n+1)} D^{n+1}F(x+dx, y+dy, z+dz\ \ldots\ldots)$$

Hence if $\theta, \theta', \theta'' \ldots\ldots$ be some quantities between 0 and 1, the remainder will be accurately represented by

$$\frac{1}{1.2.3\ \ldots\ldots\ (n+1)} D^{n+1}F(x+\theta dx, y+\theta' dy, z+\theta'' dz\ \ldots\ldots)$$

and the correct value of u' in terms of u and its differentials will be

$$u' = F(x+dx, y+dy, z+dz\ \ldots\ldots)$$

$$= F(x, y, z, \ldots) + \frac{1}{1} DF(x, y, z, \ldots) + \frac{1}{1.2} D^2F(x, y, z, \ldots) + \ldots$$

$$+ \ldots\ldots + \ldots\ldots$$

$$+ \frac{1}{1.2.3\ \ldots\ n} D^nF(x, y, z, \ldots)$$

$$+ \frac{1}{1.2.3\ \ldots\ldots\ (n+1)} D^{n+1}F(x+\theta dx, y+\theta' dy, z+\theta'' dz\ \ldots)$$

Where by the usual rules of differentiation

$$Du = \frac{du}{dx}dx + \frac{du}{dy}dy + \frac{du}{dz}dz + \ldots\ldots$$

$$D^2u = \frac{d^2u}{dx^2}dx^2 + \frac{d^2u}{dy^2}dy^2 + \frac{d^2u}{dz^2}dz^2 + \ldots\ldots$$

$$+ 2\left(\frac{d^2u}{dydz}dydz + \frac{d^2u}{dzdx}dzdx + \frac{d^2u}{dxdy}dxdy + \ldots\ldots\right)$$

$$D^3u = \ldots\ldots$$

III.

From the above formulæ we may deduce an extended form of Maclaurin's theorem; putting

$$x = 0, y = 0, z = 0 \dots .$$
$$dx = x, dy = y, dz = z \dots .$$

and

$$x\frac{d}{dx} + y\frac{d}{dy} + z\frac{d}{dz} + \dots . = \nabla$$

we have

$$F(x, y, z, \dots) = F(0) + \frac{1}{1}\nabla F(0) + \frac{1}{1.2}\nabla^2 F(0) + \dots .$$

$$+ \frac{1}{1.2.3 \dots n}\nabla^n F(0) + \frac{1}{1.2.3 \dots . (n+1)}\nabla^{n+1} F(\theta x, \theta' y, \theta'' z \dots)$$

IV.—LAGRANGE'S THEOREM.

To expand $F(y)$ in a series of ascending powers of x, where

$$y = z + xf(y) \tag{1}$$

(I.) As a first approximation, let

$$x = 0 \tag{2}$$

this gives

$$y = z \tag{3}$$

and consequently

$$F(y) = F(z) \tag{4}$$

(II.) In order to make a second approximation, let the value of y given by (3) be substituted in (1); this gives

$$y = z + xf(z) \tag{5}$$

and consequently

$$F(y) = F\{z + xf(z)\} \tag{6}$$

or by Taylor's theorem

$$F(y) = F(z) + xf(z)F'(z) \tag{7}$$

which gives the value of $F(y)$, as far as terms of the first order.

(III.) In order to make a third approximation, let the value of y given by (8) be substituted in (1); this gives

$$\left.\begin{aligned} y &= z + f\{z + xf(z)\} \\ &= z + xf(z) + x^2 f(z)f'(z) \end{aligned}\right\} \tag{8}$$

expanding by Taylor's theorem as far as terms of the second order. Consequently, instead of (7) we shall have

$$\left.\begin{aligned} F(y) &= F\{z + xf(z) + x^2 f(z)f'(z)\} \\ &= F(z) + xf(z)F'(z) + x^2 f(z)f'(z)F'(z) + \frac{x^2}{1\,.\,2}[f(z)]^2 F''(z) \end{aligned}\right\} \quad (9)$$

expanding by Taylor's theorem as far as terms of the second order. This may also be written thus,

$$F(y) = F(z) + xf(z)F'(z) + \frac{x^2}{1\,.\,2}D_z\{[f(z)]^2 F'(z)\} \quad (10)$$

(IV.) In order to make another approximation, let the value of y given by (8) be substituted in (1); this gives

$$\left.\begin{aligned} y &= z + xf\{z + xf(z) + x^2 f(z)f'(z)\} \\ &= z + xf(z) + x^2 f(z)f'(z) + x^3 f(z)(f'(z))^2 + \frac{x^3}{1\,.\,2}(f(z))^2 f''(z) \\ &= z + xf(z) + x^2 f(z)f'(z) + \frac{x^3}{1\,.\,2}D_z(f(z))^2 f'(z) \end{aligned}\right\} \quad (11)$$

expanding by Taylor's theorem as far as terms of the third order. Consequently instead of (10) we shall have

$$\left.\begin{aligned} F(y) &= F\{z + xf(z) + x^2 f(z)f'(z) + \frac{x^3}{1\,.\,2}D_z(f(z))^2 f'(z)\} \\ &= F(z) + xf(z)F'(z) + x^2 f(z)f'(z)F'(z) + \frac{x^3}{1\,.\,2}D_z[(f(z))^2 f'(z)]F'(z) \\ &\quad + \frac{x^2}{1\,.\,2}(f(z))^2 F''(z) + \frac{x^3}{1\,.\,2\,.\,3}(f(z))^3 F'''(z) + x^3(f(z))^2 f'(z)F''(z) \\ &= F(z) + xf(z)F'(z) + \frac{x^2}{1\,.\,2}D_z[(f(z))^2 F'(z)] + \frac{x^3}{1\,.\,2\,.\,3}D_z^2[(f(z))^3 F'(z)] \end{aligned}\right\} \quad (12)$$

and so on. Suppose that by continuing the approximations there had been found

$$\left.\begin{aligned} F(y) &= F(z) + \frac{x}{1}f(z)F'(z) + \frac{x^2}{1\,.\,2}D_z[(f(z))^2 F'(z)] + \frac{x^3}{1\,.\,2\,.\,3}D_z^2[(f(z))^3 F'(z)] \\ &\quad + \ldots\ldots + \frac{x^n}{1.2.3\ldots n}D_z^{n-1}[(f(z))^n F'(z)] \end{aligned}\right\} \quad (13)$$

then for the next approximation we should have the general form of the series (5), (8), or (11); viz.

$$\left.\begin{aligned} y &= z + xf(z) + x^2 f(z)f'(z) + \frac{x^3}{1\,.\,2}D_z[(f(z))^2 f'(z)] + \ldots\ldots \\ &\quad + \frac{x^{n+1}}{1.2.3\ldots n}D_z^{n-1}[(f(z))^n f'(z)] \end{aligned}\right\} \quad (14)$$

and this being substituted in (1) will give

$$\left.\begin{aligned}
F(y) = F(z) &+ \frac{x}{1}\Big\{f(z) + \frac{x}{1}f(z)f'(z) + \frac{x^2}{1.2}D_z[(f(z))^2 f'(z)] \\
&\quad + \dots + \frac{x^n}{1.2.3..n}D_z^{n-1}[(f(z))^n f'(z)]\Big\}F'(z) \\
&+ \frac{x^2}{1.2}\Big\{f(z) + \frac{x}{1}f(z)f'(z) + \frac{z^2}{1.2}D_z[(f(z))^2 f'(z)] \\
&\quad + \dots + \frac{x^n}{1.2.3..n}D_z^{n+1}[(f(z))^n f'(z)]\Big\}^2 F''(z) \\
&+ \dots\dots \\
&+ \frac{x^n}{1.2.3..n}\Big\{f(z) + \frac{x}{1}f(z)f'(z) + \frac{x^2}{1.2}D_z[(f(z))^2 f'(z)] \\
&\quad + \dots + \frac{x^n}{1.2.3\dots n}D_z^{n-1}[(f(z))^n f'(z)]\Big\}^n F^{(n)}(z) \\
&+ \frac{x^{n+}}{1.2.3..(n+1)}\Big\{f(z) + \frac{x}{1}f(z)f'(z) + \frac{x^2}{1.2}D_z[(f(z))^2 f'(z)] \\
&\quad + \dots + \frac{x^n}{1.2.3\dots n}D_z^{n-1}[(f(z)\ f'(z)]\Big\}^{n+1} F^{(n+1)}(z)
\end{aligned}\right\}\quad(15)$$

or writing for convenience

$$f(z) = u$$

$$\left.\begin{aligned}
F(y)=F(z) &+ \frac{x}{1}\left\{u + x\frac{du^2}{1.2} + x^2\frac{d^2u^3}{1.2.3} + \dots + x^n\frac{d^n u^{n+1}}{1.2.3..(n+1)}\right\}F'(z) \\
&+ \frac{x^2}{1.2}\left\{u + x\frac{du^2}{1.2} + x^2\frac{d^2u^3}{1.2.3} + \dots + x^n\frac{d^n u^{n+1}}{1.2.3..(n+1)}\right\}^2 F''(z) \\
&+ \dots\dots \\
&+ \frac{x^n}{1.2.3\dots n}\left\{u + x\frac{du^2}{1.2} + x^2\frac{d^2u^3}{1.2.3} + \dots + x^n\frac{d^n u^{n+1}}{1.2.3..(n+1)}\right\}^n F^{(n)}(z) \\
&+ \frac{x^{n+1}}{1.2.3..(n+1)}\left\{u + x\frac{du^2}{1.2} + x^2\frac{d^2u^3}{1.2.3} + .. + x^n\frac{d^n u^{n+1}}{1.2.3..(n+1)}\right\}^{n+1} F^{(n+1)}(z)
\end{aligned}\right\}(16)$$

But writing for convenience

$$u + x\frac{du^2}{1.2} + x^3\frac{d^2u^3}{1.2.3} + \dots = \Omega \tag{17}$$

it will be found by ordinary processes that

$$u^{n+1} = u^{n+1}$$

$$du^{n+1} = (n+1)u^n du$$

$$= \frac{n+1}{n} \times \text{coefficient of } x \text{ in the expansion of } \Omega^n$$

$$d^2u^{u+1} = (n+1)\{nu^{n-1}(du)^2 + u^n d^2u\}$$

$$= \frac{n+1}{\frac{n-1}{1.2}} \times \text{ coefficient of } x^2 \text{ in the expansion of } \Omega^{n-1}$$

$$d^3u^{n+1} = (n+1)\{n(n-1)u^{n-2}(du)^3 + 3nu^{n-1}dud^2u + u^n d^3u\}$$

$$= \frac{n+1}{\frac{n-2}{1.2.3}} \times \text{ coefficient of } x^3 \text{ in the expansion of } \Omega^{n-2}$$

. . . .

so that the coefficient of $\dfrac{x^{n+1}}{1.2.3\ldots n(n+1)}$ in the $(n+1)$th line of (15) will be

$$F^{(n+1)}(z)(f(z))^{n+1}$$

. . . . in the nth

$$\frac{n}{1}F^{(n)}(z)D_z(f(z))^{n+1}$$

. . . .

. . . . in the 3d

$$\frac{n(n-1)}{1.2}F'''(z)D_z^{n-2}(f(z))^{n+1}$$

. . . . in the 2d

$$\frac{n}{1}F''(z)D_z^{n-1}(f(z))^{n+1}$$

. . . . in the 1st

$$F'(z)D_z^n(f(z))^{n+1}$$

consequently the whole coefficient of $\dfrac{x^{n+1}}{1.\,2.\,3.\,\ldots\,(n+1)}$ in the expression for $F(y)$ will be

$$\left.\begin{aligned} &F'(z)D_z^n(f(z))^{n+1} + \frac{n}{1}F''(z)D_z^{n-1}(f(z))^{n+1} + \frac{n.(n-2)}{1.2}F'''(z)D_z^{n-2}(f(z))^{n+1} + \ldots \\ &\qquad + \frac{n}{1}F^{(n)}(z)D_z(f(z))^{n+1} + F^{(n+1)}(z)(f(z))^{n+1} \end{aligned}\right\} \quad (18)$$

which by Leibnitz's theorem

$$= D_z^n\{(f(z))^{n+1}F^1(z)\} \qquad (19)$$

so that the series (13) is generally true whatever be the value of n.

V.—VANISHING FRACTIONS.

If u be a function of x of the form

$$u = \frac{\mathrm{f}(x)}{f(x)} \tag{1}$$

where $\mathrm{f}(x)$ and $f(x)$ are any functions of x, which remain continuous between the limits

$$\left.\begin{array}{l} x = a \text{ and } x = a + \varepsilon \\ x = a \text{ and } x = a - \varepsilon \end{array}\right\} \tag{2}$$

exclusive of the first of each pair of values, it sometimes happens that for certain values of the variable x, such as

$$x = a \tag{3}$$

the functions $\mathrm{f}(x), f(x)$ sensibly vanish; so that, if U be the value of u when (3) is satisfied,

$$\mathrm{U} = \frac{\mathrm{f}(a)}{f(a)} = \frac{0}{0} \tag{4}$$

and the value of U is consequently indeterminate; the question then arises how the value of U is to be found. Now, according to the present theory of infinitesimals, the values of $\mathrm{f}(a), f(a)$ will not in general be absolutely zero, but merely infinitely small, so that the solution of the problem depends upon the determination of the orders to which the functions in question respectively belong. If the order of $\mathrm{f}(a)$ be higher than that of $f(a)$ the value of U will evidently be infinitely small; but if the order of $f(a)$ be higher than that of $\mathrm{f}(a)$ the value of U will be infinitely great; in fact, if $\mathrm{f}(a)$ be of the order m, and $f(a)$ of the order n, U will be of the order

$$m - n$$

and this will give rise to three cases,

(1) If $m > n$, $(m-n)$ is positive, and U will be infinitely small,

(2) If $m < n$, $(m-n)$ is negative, and U will be infinitely great,

(3) If $m = n$, $(m-n) = 0$, and U will be finite.

Now, since the functions $\mathrm{f}(x)$, $f(x)$ remain continuous between the limits determined by the equations (2), an approximation may be made to the values of $\mathrm{f}(a), f(a)$, by assigning to x some value a little greater or less than a contained between the above-mentioned limits. If then

ε be an infinitesimal of the first order at least, the value of U will be given nearly by one of the equations.

$$U = \frac{\mathrm{f}(a+\varepsilon)}{f(a+\varepsilon)} \text{ or } U = \frac{\mathrm{f}(a-\varepsilon)}{f(a-\varepsilon)} \tag{5}$$

Now, by Taylor's theorem,

$$\left.\begin{aligned} \mathrm{f}(a+\varepsilon) &= \mathrm{f}(a) + \frac{\varepsilon}{1}\mathrm{f}'(a) + \frac{\varepsilon^2}{1.2}\mathrm{f}''(a) + \dots \\ f(a+\varepsilon) &= f(a) + \frac{\varepsilon}{1}f'(a) + \frac{\varepsilon^2}{1.2}f''(a) + \dots \end{aligned}\right\} \tag{6}$$

and

$$\left.\begin{aligned} \mathrm{f}(a-\varepsilon) &= \mathrm{f}(a) - \frac{\varepsilon}{1}\mathrm{f}'(a) + \frac{\varepsilon^2}{1.2}\mathrm{f}''(a) - \dots \\ f(a-\varepsilon) &= f(a) - \frac{\varepsilon}{1}f'(a) + \frac{\varepsilon^2}{1.2}f''(a) - \dots \end{aligned}\right\} \tag{7}$$

Hence if these values be substituted for $\mathrm{f}(a+\varepsilon)$, $f(a+\varepsilon)$, $\mathrm{f}(a-\varepsilon)$, $f(a-\varepsilon)$ respectively in (5), we shall have (remembering that by the hypothesis

$$\mathrm{f}(a) = 0 \qquad f(a) = 0) \tag{8}$$

$$U = \frac{\mathrm{f}'(a) + \frac{\varepsilon}{2}\mathrm{f}''(a) + \dots}{f'(a) + \frac{\varepsilon}{2}f''(a) + \dots} \tag{9}$$

or

$$U = \frac{\mathrm{f}'(a) - \frac{\varepsilon}{2}\mathrm{f}''(a) + \dots}{f'(a) - \frac{\varepsilon}{2}f''(a) + \dots} \tag{10}$$

from both of which expressions there would result, if infinitesimals be neglected in comparison with finite ones,

$$U = \frac{\mathrm{f}'(a)}{f'(a)} \tag{11}$$

If the first derived functions themselves vanished, and in consequence of the conditions

$$\mathrm{f}'(a) = 0 \qquad f'(a) = 0 \tag{12}$$

the expression (11) became indeterminate, it would be necessary to take in another term in the expressions (6) or (7), in which case (9) or (10) combined with (8) and (12) would give

$$U = \frac{\mathrm{f}''(a)}{f'(a)} = 0 \tag{13}$$

And generally, if the derived functions up to the $(n-1)$th vanished together with the functions themselves, the value of U would be given by the equation

$$U = \frac{\mathrm{f}^{(n)}(a)}{f^{(n)}(a)} \tag{14}$$

It may here be remarked that the conditions (8) and (12) and generally the conditions

$$\mathrm{f}^{(i)}(a) = 0 \qquad f^{(i)}(a) = 0 \tag{15}$$

represent (as is known by the theory of equations) that the equations (8) have each $(i+1)$ equal roots, a factor corresponding to each of which is expelled by each differentiation; so that the degree of the factor (which in the present case will be an infinitesimal one) is continually reduced by unity, until when it is integral, it at length becomes actually 0; in which case, if there be no other infinitesimal factors, $\mathrm{f}^{(i+1)}(a)$, $f^{(i+1)}(a)$ have finite values, and the value of the fraction is determinate.

Sometimes, however, it happens that the index of the degree of the infinitesimal factor is fractional, so that, although it is continually decreased by unity, it never vanishes, but passes from positive to negative; so that the fraction always appears under one of the indeterminate forms

$$\frac{0}{0} \quad \text{or} \quad \frac{\infty}{\infty} \tag{16}$$

and its value consequently cannot be determined by the above process. Suppose then that the order of the infinitesimal factor is fractional and $= \frac{m}{n}$, and suppose that

$$\frac{m}{n} = p + \frac{q}{n} \tag{17}$$

where p is a whole number. Then by p ordinary differentiations the order of the factor will be reduced from $\frac{m}{n}$ to $\frac{q}{n}$, and it is then required to reduce an infinitesimal of the order $\frac{q}{n}$ to a finite number, or in other words to decrease the order of the infinitesimal by the $\frac{q}{n}$th part of unity; this must be done, as was seen in a former paper (on Infinitesimal Analysis), by choosing ϵ of an order $= \frac{q}{n}$. But as it was seen in the same paper that this can scarcely be called practicable, the general formulæ for

$$\frac{d_r\,\mathrm{f}x}{dx^r} \quad \text{and} \quad \frac{d^r f(x)}{dx^r} \tag{18}$$

must be found, and afterwards r must be put equal to $\frac{q}{n}$. This, according to the principles of infinitesimal analysis, will produce the same result.

VI.

Def. A plane is a sphere whose radius is infinitely great. It is proposed to find the analytical expression for a plane according to the above definition. Let the equation to the sphere be

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2 \tag{1}$$

or

$$x^2 + y^2 + z^2 - 2(ax+by+cz) + a^2 + b^2 + c^2 = r^2 \tag{2}$$

Now in this expression it is obvious that a, b, c, the co-ordinates of the centre of the sphere, are of the same order as r. Hence dividing by r^2

$$\frac{a^2+b^2+c^2}{r^2} - \frac{2}{r^2}(ax+by+cz) + \frac{x^2+y^2+z^2}{r^2} = 1 \tag{3}$$

which, if we neglect infinitesimals of the second order, becomes

$$ax + by + cz = \frac{1}{2}(a^2+b^2+c^2-r^2) \tag{4}$$

or, writing for convenience

$$a^2 + b^2 + c^2 = k^2 \tag{5}$$

$$\frac{a}{k}x + \frac{b}{k}y + \frac{c}{k}z = \frac{1}{2k}(k^2-r^2). \tag{6}$$

Now the second side of this equation may also be written thus

$$\frac{1}{2}(k-r)\left(1+\frac{r}{k}\right)$$

and r and k are approximately equal to one another, because the origin is supposed to be at a finite distance from the surface; also $(k-r)$ is the shortest of the two perpendiculars from the origin upon the surface, and if we represent it by p (6) may be thus written

$$\frac{a}{k}x + \frac{b}{k}y + \frac{c}{k}z = p \tag{8}$$

in which also $\frac{a}{k}, \frac{b}{k}, \frac{c}{k}$, are the direction cosines of the line joining the origin and the centre of the sphere, and consequently also of the perpendicular (p) on the plane. This equation may also be put under another form, for if α, β, γ be the co-ordinates of the point in which p meets the surface

$$p = l\alpha + m\beta + n\gamma \tag{9}$$

where

$$l = \frac{a}{k}, \ m = \frac{b}{k}, \ n = \frac{c}{k} \tag{10}$$

and consequently (8) becomes

$$l(x-\alpha) + m(y-\beta) + n(z-\gamma) = 0 \tag{11}$$

If the origin be taken on the surface we have

$$\alpha = 0, \beta = 0, \gamma = 0 \tag{12}$$

and consequently

$$lx + my + nz = 0 \tag{13}$$

the equation to a plane passing through the origin.

There is one other form of the equation worthy of notice, which is as follows; dividing (8) by p throughout we have

$$\frac{x}{\frac{kp}{a}} + \frac{y}{\frac{kp}{b}} + \frac{z}{\frac{kp}{c}} = 1 \tag{14}$$

or writing for convenience

$$f = \frac{kp}{a}, \quad g = \frac{kp}{b}, \quad h = \frac{kp}{c} \tag{15}$$

$$\frac{x}{f} + \frac{y}{g} + \frac{z}{h} = 1 \tag{16}$$

in which f, g, h are evidently the projections of the perpendicular p on the three co-ordinate axes respectively, or the intercepts of those axes between the origin and the plane.

VII.

Def. A point is a sphere whose radius is infinitely small. In this case we may neglect r, and consequently r^2; hence (1) becomes

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = 0 \tag{17}$$

which is satisfied only by

$$x = a,\ y = b,\ z = c \tag{18}$$

These are, therefore, the analytical expressions for a point. If the origin be taken at the point, we have

$$a = 0,\ b = 0,\ c = 0 \tag{19}$$

and consequently

$$x = 0,\ y = 0,\ z = 0 \tag{20}$$

VIII.

Def. Two straight lines are said to be parallel when their point of intersection is at an infinite distance.

Among the various ways in which the analytical expression for this definition may be found the two following appear worthy of notice.

Suppose first that the two lines be not nearly parallel to any of the three co-ordinate axes.

Let the equations to the two straight lines be

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} \tag{1}$$

$$\frac{x-a'}{l'} = \frac{y-b'}{m'} = \frac{z-c'}{n'} \tag{2}$$

in which expressions x, y, z are the co-ordinates of the point of intersection; combining (1) and (2) we have

$$\frac{x-a}{x-a'}\frac{l'}{l} = \frac{y-b}{y-b'}\frac{m'}{m} = \frac{z-c}{z-c'}\frac{n'}{n} \tag{3}$$

which may also be written thus

$$\frac{1-\frac{a}{x}}{1-\frac{a'}{x}}\frac{l'}{l} = \frac{1-\frac{b}{y}}{1-\frac{b'}{y}}\frac{m'}{m} = \frac{1-\frac{c}{z}}{1-\frac{c'}{z}}\frac{n'}{n} \tag{4}$$

If then the point of intersection be removed to an infinite distance, the co-ordinates x, y, z will become infinitely great, and the ratios

$$\frac{a}{x} \quad \frac{a'}{x} \quad \frac{b}{y} \quad \frac{b'}{y} \quad \frac{c}{z} \quad \frac{c'}{z}$$

infinitely small; they may consequently be neglected in the expressions (4), and the conditions of parallelism become

$$\frac{l'}{l} = \frac{m'}{m} = \frac{n'}{n} \tag{5}$$

or

$$l = l', \quad m = m', \quad n = n' \tag{6}$$

as usual.

If the two lines be so placed that one of the co-ordinates of the point of intersection does not become infinite, the above method fails; and although by a transformation of co-ordinates the difficulty might be obviated, yet it will be simpler to treat the question thus: the equations to the two lines may be written in the following form,

$$x = a + lr \qquad y = b + mr \qquad z = c + nr \tag{7}$$

$$x = a' + l'r' \qquad y = b' + m'r' \qquad z = c' + n'r' \tag{8}$$

in which expressions, r, r' are the distances of the point of intersection from the points of a, b, c and a', b', c' respectively. Subtracting (8) from (7) we have

$$\left.\begin{aligned} a - a' + lr - l'r' &= 0 \\ b - b' + mr - m'r' &= 0 \\ c - c' + nr - n'r' &= 0 \end{aligned}\right\} \tag{9}$$

or, neglecting a, b, c, a', b', c', in comparison with r and r', we find

$$\frac{l}{l'} = \frac{m}{m'} = \frac{n}{n'} \tag{10}$$

as before.

IX.—TO FIND AN EXPRESSION FOR THE RADIUS OF ABSOLUTE CURVATURE IN CURVES OF DOUBLE CURVATURE.

Def. The circle of absolute curvature is a circle which passes through three consecutive points of the curve.

If the co-ordinates of the first point on the curve through which the circle of curvature is to pass are

$$x, y, z \tag{1}$$

those of the consecutive point will be

$$x + dx \qquad y + dy \qquad z + dz \tag{2}$$

and those of the third

$$\left.\begin{aligned} x + dx + d(x+dx) &= x + 2dx + d^2x \\ y + dy + d(y+dy) &= y + 2dy + d^2y \\ z + dz + d(z+dz) &= z + 2dz + d^2z \end{aligned}\right\} \tag{3}$$

Also let the equation to the sphere of which the circle is a section be

$$(x-a)^2 + (y-\beta)^2 + (z-\gamma)^2 = \rho^2 \tag{4}$$

then since it also passes through the consecutive point, the equation (4) must be satisfied by the co-ordinates (2), in other words the following equation must also hold good,

$$(x+dx-a)^2 + (y+dy-\beta)^2 + (z+dz-\gamma)^2 = \rho^2 \tag{5}$$

and similarly also the following one

$$(x+2dx+d^2x-a)^2 + (y+2dy+d^2y-\beta)^2 + (z+2dz+d^2z-\gamma)^2 = \rho^2 \tag{6}$$

Expanding (4) and (5), and subtracting one from the other, there results, and writing for convenience

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{7}$$

there results

$$(x-a)\frac{dx}{ds} + (y-\beta)\frac{dy}{ds} + (z-\gamma)\frac{dz}{ds} = -\frac{ds}{2} \tag{8}$$

or, neglecting infinitesimal quantities,

$$(x-a)\frac{dx}{ds} + (y-\beta)\frac{dy}{ds} + (z-\gamma)\frac{dz}{ds} = 0 \tag{9}$$

Also, expanding (4) and (6), and combining (8) with the result, we have

$$\left.\begin{aligned}&(x-a)\frac{d^2x}{ds^2}+(y-\beta)\frac{d^2y}{ds^2}+(z-\gamma)\frac{d^2z}{ds^2}\\ &=-1-2(dxd^2x+dyd^2y+dzd^2z)-\tfrac{1}{2}\{(d^2x)^2+(d^2y)^2+(d^2z)^2\}\end{aligned}\right\}\qquad(10)$$

or neglecting infinitesimals of the third and higher orders,

$$(x-a)\frac{d^2x}{ds^2}+(y-\beta)\frac{d^2y}{ds^2}+(z-\gamma)\frac{d^2z}{ds^2}=-1\qquad(11)$$

Again, since the circle is a plane curve its co-ordinates must satisfy the equation to a plane as well as the equations (4), (5), (6); let this be

$$l(x-a)+m(y-\beta)+n(z-\gamma)=0\qquad(12)$$

then since this plane passes through the points (2) and (3) also the equation (11) must be satisfied by the co-ordinates of those points; hence, besides (11) the two following must also hold good,

$$l(x+dx-a)+m(y+dy-\beta)+n(z+dz-\gamma)=0\qquad(13)$$

$$l(x+2dx+d^2x-a)+m(y+2dy+d^2y-\beta)+n(z+2dz+d^2z-\gamma)=0\qquad(14)$$

The combination of which last three equations produces the two following

$$ldx+mdy+ndz=0\qquad(15)$$

$$ld^2x+md^2y+nd^2z=0\qquad(16)$$

eliminating l, m, n, in turn from these, we find

$$\frac{l}{dyd^2z-dzd^2y}=\frac{m}{dzd^2x-dxd^2z}=\frac{n}{dxd^2y-dyd^2x}\qquad(17)$$

eliminating l, m, n, between which and (11), we have

$$(x-a)(dyd^2z-dzd^2y)+(y-\beta)(dzd^2x-dxd^2z)+(z-\gamma)(dxd^2y-dyd^2x)=0\quad(18)$$

Hence for the determination of the co-ordinates of the centre of the circle, that is, the quantities $x-\alpha$, $y-\beta$, $z-\gamma$, and the radius ρ, there exist the three equations

$$\left.\begin{aligned}&(x-a)\frac{dx}{ds}+(y-\beta)\frac{dy}{ds}+(z-\gamma)\frac{dz}{ds}=0\\ &(x-a)\frac{d^2x}{ds^2}+(y-\beta)\frac{d^2y}{ds^2}+(z-\gamma)\frac{d^2z}{ds^2}=-1\\ &(x-a)\mathrm{A}+(y-\beta)\mathrm{B}+(z-\gamma)\mathrm{C}=0\end{aligned}\right\}\qquad(19)$$

where

$$\left.\begin{aligned}\mathrm{A}&=\frac{dy}{ds}\frac{d^2z}{ds^2}-\frac{dz}{ds}\frac{d^2y}{ds^2}\\ \mathrm{B}&=\frac{dz}{ds}\frac{d^2x}{ds^2}-\frac{dx}{ds}\frac{d^2z}{ds^2}\\ \mathrm{C}&=\frac{dx}{ds}\frac{d^2y}{ds^2}-\frac{dy}{ds}\frac{d^2x}{ds^2}\end{aligned}\right\}\qquad(20)$$

hence, in writing for convenience,

$$\Omega^2 = A^2 + B^2 + C^2 \tag{21}$$

we find by cross-multiplication from (19)

$$\left.\begin{aligned} \Omega^2(x-a) &= B\frac{dz}{ds} - C\frac{dy}{ds} \\ \Omega^2(y-\beta) &= C\frac{dx}{ds} - A\frac{dz}{ds} \\ \Omega^2(z-\gamma) &= A\frac{dy}{ds} - B\frac{dx}{ds} \end{aligned}\right\} \tag{22}$$

But since by (7)

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1 \tag{23}$$

consequently

$$\left(\frac{d(x+dx)}{d(s+ds)}\right)^2 - \left(\frac{dx}{ds}\right)^2 + \left(\frac{d(y+dy)}{d(z+dz)}\right)^2 - \left(\frac{dy}{ds}\right)^2 + \left(\frac{d(z+dz)}{d(s+ds)}\right)^2 - \left(\frac{dz}{ds}\right)^2 = 0 \tag{24}$$

But

$$\frac{K}{(d(s+ds))^2} = \frac{K}{ds^2 + 2dsd^2s + (d^2s)^2} = \frac{K}{ds^2}\left\{1 - \frac{2}{ds^2}(2dsd^2s + (d^2s)^2) + \ldots\right\} \tag{25}$$

or neglecting infinitesimal quantities, or equating to zero the second differential of s, which will come to the same thing, since s may be taken as the independent variable,

$$\frac{K}{(d(s+ds))^2} = \frac{K}{ds^2} \tag{26}$$

hence neglecting also

$$(d^2x)^2 \qquad (d^2y)^2 \qquad (d^2z)^2$$

in comparison with

$$dxd^2x \qquad dyd^2y \qquad dzd^2z$$

(24) may finally be written

$$\frac{dx}{ds}\frac{d^2x}{ds^2} + \frac{dy}{ds}\frac{d^2y}{ds^2} + \frac{dz}{ds}\frac{d^2z}{ds^2} = 0 \tag{27}$$

Hence (22) are reduced to

$$\Omega^2(x-a) = \frac{d^2x}{ds^2} \qquad \Omega^2(y-\beta) = \frac{d^2y}{ds^2} \qquad \Omega^2(z-\gamma) = \frac{d^2z}{ds^2} \tag{28}$$

but since, also, by means of adding and subtracting the quantities

$$\left(\frac{dx}{ds}\frac{d^2x}{ds^2}\right)^2 \qquad \left(\frac{dy}{ds}\frac{d^2y}{ds^2}\right)^2 \qquad \left(\frac{dz}{ds}\frac{d^2z}{ds^2}\right)^2$$

and keeping the condition (28) in mind, it is found that

$$\Omega^2 = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2 \tag{29}$$

(28) may also be written in the following form,

$$\frac{x-a}{\frac{d^2x}{ds^2}} = \frac{y-\beta}{\frac{d^2y}{ds^2}} = \frac{z-\gamma}{\frac{d^2z}{ds^2}} = \frac{1}{\Omega^2} = \frac{\rho}{\Omega} \tag{30}$$

hence also

$$\frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right) + \left(\frac{d^2z}{ds^2}\right)^2 \tag{31}$$

It may here be remarked, that by means of dividing throughout by ds^2 we have been enabled to neglect terms involving d^2s; if, however, this had not been done, a formula somewhat similar would have been found, as follows: instead of (19) we should have had

$$\left.\begin{array}{l}(x-a)dx + (y-\beta)dy + (z-\gamma)dz = 0 \\ (x-a)d^2x + (y-\beta)d^2y + (z-\gamma)d^2z = -\,ds^2 \\ (x-a)\mathrm{P} + (y-\beta)\mathrm{Q} + (z-\gamma)\mathrm{R} = 0\end{array}\right\} \tag{32}$$

where

$$\mathrm{P} = dyd^2z - dzd^2y \qquad \mathrm{Q} = dzd^2x - dxd^2z \qquad \mathrm{R} = dxd^2y - dyd^2x \tag{33}$$

which would have given

$$\left.\begin{array}{l}\Omega'^2(x-a) = ds^2(\mathrm{Q}dz-\mathrm{R}dy) = \{ds^2d^2x - dx(dxd^2x+dyd^2y+dzd^2z)\}ds^2 \\ \Omega'^2(y-\beta) = ds^2(\mathrm{R}dx-\mathrm{P}dz) = \{ds^2d^2y - dy(dxd^2x+dyd^2y+dzd^2z)\}ds^2 \\ \Omega'^2(z-\gamma) = ds^2(\mathrm{P}dy-\mathrm{Q}dx) = \{ds^2d^2z - dz(dxd^2x+dyd^2y+dzd^2z)\}ds^2\end{array}\right\} \tag{34}$$

where

$$\Omega'^2 = \mathrm{P}^2 + \mathrm{Q}^2 + \mathrm{R}^2 \tag{35}$$

But

$$(d(x+dx))^2 + (d(y+dy))^2 + (d(z+dz))^2 = (d(s+ds))^2 \tag{36}$$

hence expanding and subtracting (7) from the result, and neglecting infinitesimal quantities of higher orders in comparison with those of lower, we find

$$dxd^2x + dyd^2y + dzd^2z = dsd^2s \tag{37}$$

and consequently (34) may be written also thus,

$$\left.\begin{array}{l}\Omega'^2(x-a) = (ds^2d^2x - dxdsd^2s)ds^2 \\ \Omega'^2(y-\beta) = (ds^2d^2y - dydsd^2s)ds^2 \\ \Omega'^2(z-\gamma) = (ds^2d^2z - dzdsd^2s)ds^2\end{array}\right\} \tag{38}$$

Now from (34)

$$\left.\begin{array}{rl}\rho(\mathrm{P}^2+\mathrm{Q}^2+\mathrm{R}^2) &= ds^2\{(\mathrm{Q}dz-\mathrm{R}dy)^2 + (\mathrm{R}dx-\mathrm{P}dz)^2 + (\mathrm{P}dy-\mathrm{Q}dx)^2\}^{\frac{1}{2}} \\ &= ds^2\{(\mathrm{P}^2+\mathrm{Q}^2+\mathrm{R}^2)ds^2 - (\mathrm{P}dx+\mathrm{Q}dy+\mathrm{R}dz)^2\}^{\frac{1}{2}}\end{array}\right\} \tag{39}$$

But

$$Pdx + Qdy + Rdz = 0 \tag{40}$$

identically ; hence

$$\left.\begin{aligned} \rho &= \frac{ds^3}{(P^2+Q^2+R^2)^{\frac{1}{2}}} \\ &= \frac{ds^3}{\{(dyd^2z-dzd^2y)^2 + (dzd^2x-dxd^2z)^2 + (dxd^2y-dyd^2x)^2\}^{\frac{1}{2}}} \end{aligned}\right\} \tag{41}$$

Again, from (38)

$$\Omega'^2\rho = ds^3\{(dsd^2x-dxd^2s)^2 + (dyd^2y-dsd^2s)^2 + (dsd^2z-dzd^2s)^2\}^{\frac{1}{2}} \tag{42}$$

or by means of (42)

$$\left.\begin{aligned} \frac{ds^3}{\rho} &= \{(dsd^2x-dxd^2s)^2 + (dsd^2y-dyd^2s)^2 + (dsd^2z-dzd^2s)^2\}^{\frac{1}{2}} \\ &= \{ds^2[(d^2x)^2 + (d^2y)^2 + (d^2z)^2] - dsd^2s(dxd^2x+dyd^2y+dzd^2z\}^{\frac{1}{2}} \\ &= ds\{(d^2x)^2 + (d^2y)^2 + (d^2z)^2 - (d^2s)^2\}^{\frac{1}{2}} \end{aligned}\right\} \tag{43}$$

hence also

$$\rho = \frac{ds^2}{\{(d^2x)^2 + (d^2y)^2 + (d^2z)^2-(d^2s)^2\}^{\frac{1}{2}}} \tag{44}$$

Again, (38) by means of (41) gives

$$\left.\begin{aligned} x - a &= ds^3\frac{dsd^2x - dxd^2s}{P^2 + Q^2 + R^2} = \rho^2\frac{d\left(\frac{dx}{ds}\right)}{ds} \\ y - \beta &= ds^3\frac{dsd^2y - dyd^2s}{P^2 + Q^2 + R^2} = \rho^2\frac{d\left(\frac{dy}{ds}\right)}{ds} \\ z - \gamma &= ds^3\frac{dsd^2z - dzd^2s}{P^2 + Q^2 + R^2} = \rho^2\frac{d\left(\frac{dz}{ds}\right)}{ds} \end{aligned}\right\} \tag{45}$$

So that if a, b, c be the angles which the radius of curvature makes with the three co-ordinate axes respectively

$$\left.\begin{aligned} \text{Cos } a &= \frac{x-a}{\rho} = \rho\frac{d\left(\frac{dx}{ds}\right)}{ds} \\ \text{Cos } b &= \frac{y-\beta}{\rho} = \rho\frac{d\left(\frac{dy}{ds}\right)}{ds} \\ \text{Cos } c &= \frac{z-\gamma}{\rho} = \rho\frac{d\left(\frac{dz}{ds}\right)}{ds} \end{aligned}\right\} \tag{46}$$

and also

$$\frac{ds}{\rho} = \left\{ \left(d\frac{dx}{ds}\right)^2 + \left(d\frac{dy}{ds}\right)^2 + \left(d\frac{dz}{ds}\right)^2 \right\}^{\frac{1}{2}} \tag{47}$$

and if d^2s be made to vanish either (38) and (44) or (46) and (47) will give

$$\operatorname{Cos} a = \rho\frac{d^2x}{ds^2} \qquad \operatorname{Cos} b = \rho\frac{d^2y}{ds^2} \qquad \operatorname{Cos} c = \rho\frac{d^2z}{ds^2} \tag{48}$$

as before.

The above are the principal formulæ in use for determining the magnitude and direction of the radius of curvature of curves of double curvature. The corresponding formulæ for plane curves are easily deducible from those given above by omitting terms involving z and its differentials.

X.—MAXIMA AND MINIMA.

When a function of any number of variables

$$x, y, z, \ldots$$

receives a particular real value which is greater than all neighbouring real values, that is, all values obtained by increasing or decreasing the values of x, y, x, . . . by infinitely small quantities, that particular value is called a maximum. When the particular value of the function is real and less than all neighbouring real values, it is called a minimum.

Thus, if

$$u = f(x, y, z, \ldots) \tag{1}$$

the condition that u may be a maximum will be that both

$$f(x+dx,\ y+dy,\ z+dz,\ \ldots) - f(x, y, z, \ldots) \tag{2}$$

and

$$f(x-dx,\ y-dy,\ z-dz,\ \ldots) - f(x, y, z, \ldots) \tag{3}$$

be negative, or expanding by Taylor's Theorem,

$$\mathrm{D}u + \frac{1}{1.2}\ \mathrm{D}^2u + \frac{1}{1.2.3}\ \mathrm{D}^3u + \ldots \tag{4}$$

and

$$-\mathrm{D}u + \frac{1}{1.2}\ \mathrm{D}^2u - \frac{1}{1.2.3}\ \mathrm{D}^3u + \ldots \tag{5}$$

must both be negative. But since the orders of infinitesimals to which the various terms of these series belong decrease by unity as we pass from left to right, the sum of all the terms after the first can have no effect on the first term; hence in order that the condition may be fulfilled the first term of (4) and (5) must be negative. This can evidently be the case only when

$$\mathrm{D}u = 0 \tag{6}$$

So likewise, if when (6) is satisfied the second term of (4) and (5) also vanish, it will be necessary that

$$\mathrm{D}^3 u = 0 \tag{7}$$

and so generally if the $2n$th differential of u vanishes it will be necessary that

$$\mathrm{D}^{2n+1} u = 0 \tag{8}$$

A similar train of reasoning would show that if u is to be a minimum the series (4) and (5) must both be positive; this will obviously lead to the same system of conditions (6), (7), (8), as in the former case.

Hence if any system of values of the variables $x, y, z, \ldots$ make

$$\mathrm{D}^{2n+1} u = 0 \tag{9}$$

and

$$\mathrm{D}^{2(n+1)} u = \text{negative quantity} \tag{10}$$

u will be a maximum. If on the other hand any system of values of the variables $x, y, z, \ldots$ make

$$\mathrm{D}^{2n+1} u = 0 \tag{11}$$

and

$$\mathrm{D}^{2(n+1)} u = \text{positive quantity} \tag{12}$$

u will be a minimum. If however it happens that as often as

$$\mathrm{D}^{2n+1} u = 0 \tag{13}$$

also

$$\mathrm{D}^{2(n+1)} = 0 \tag{14}$$

then u will admit of neither a maximum or a minimum value.

From these considerations are to be deduced the conditions which the differential coefficients of Ω must satisfy independently of the differentials

$$dx, dy, dz, \ldots \tag{15}$$

Now adopting a common notation

$$\left.\begin{aligned} \mathrm{D}^2\Omega = {} & (udx + w'dy + v'dz + \ldots)dx \\ & + (w'dx + vdy + u'dz + \ldots)dy \\ & + (v'dx + u'dy + wdz + \ldots)dz \\ & + \ldots\ldots \end{aligned}\right\} \tag{16}$$

which expression may be written also in any one of the n different ways

$$\left.\begin{array}{l} udx^2 + 2(v'dz + w'dy + \ldots)dx + vdy^2 + wdz^2 + \ldots + 2(u'dydz + \ldots) \\ vdy^2 + 2(w'dx + u'dz + \ldots)dy + wdz^2 + udx^2 + \ldots + 2(v'dzdx + \ldots) \\ wdz^2 + 2(u'dy + v'dx + \ldots)dz + udx^2 + vdy^2 + \ldots + 2(w'dxdy + \ldots) \\ \ldots\ldots\ldots \end{array}\right\} \tag{17}$$

Now in order that the requisite conditions may be satisfied, we shall derive as a condition from these expressions respectively

$$\left.\begin{array}{l}(v'dz+w'dy+\ldots)^2-u[vdy^2+wdz^2+\ldots+2(u'dydz+\ldots)]<0\\(w'dx+u'dz+\ldots)^2-v[wdz^2+udx^2+\ldots+2(v'dzdx+\ldots)]<0\\(u'dy+v'dx+\ldots)^2-w[udx^2+vdy^2+\ldots+2(w'dxdy+\ldots)]<0\\\ldots\ldots\ldots\end{array}\right\}\quad(18)$$

which may be also written thus

$$\left.\begin{array}{l}(v'^2-wu)dx^2+2[(u'v'-ww')dy+\ldots]dx+(u'^2-vw)dy^2+\ldots<0\\(w'^2-uv)dy^2+2[(v'w'-uu')dz+\ldots]dy+(v'^2-wu)dz^2+\ldots<0\\(u'^2-vw)dz^2+2[(w'u'-vv')dx+\ldots]dz+(w'^2-uv)dx^2+\ldots<0\\\ldots\ldots\ldots\end{array}\right\}\quad(19)$$

Similarly these would give as the condition

$$\left.\begin{array}{l}[(u'v'-ww')dy+\ldots]^2-(v'^2-wu)[(u'^2-vw)dy^2+\ldots]<0\\{}[(v'w'-uu')dz+\ldots]^2-(w'^2-uv)[(v'^2-wu)dz^2+\ldots]<0\\{}[(w'u'-vv')dx+\ldots]^2-(u'^2-vw)[(w'^2-uv)dx^2+\ldots]<0\\\ldots\ldots\ldots\end{array}\right\}\quad(20)$$

and so on until by successive reduction we should at length obtain a result independent of the differentials.

In the case of two variables (x, y) (18) would give as the criterion

$$w'^2-uv<0 \qquad (21)$$

or

$$\left(\frac{d^2\Omega}{dxdy}\right)^2-\frac{d^2\Omega}{dx^2}\,\frac{d^2\Omega}{dy^2}<0 \qquad (22)$$

In the case of three variables $(x, y, z,)$ (20) would give as the criteria

$$\left.\begin{array}{l}(v'w'-uu')^2-(v'^2-wu)(w'^2-uv)<0\\(w'u'-vv')^2-(w'^2-uv)(u'^2-vw)<0\\(u'v'-ww')^2-(u'^2-vw)(v'^2-wu)<0\end{array}\right\}\quad(23)$$

or, as they may be also written

$$\left.\begin{array}{l}u(uvw-uu'^2-vv'^2-ww'^2+2u'v'w')>0\\v(uvw-uu'^2-vv'^2-ww'^2+2u'v'w')>0\\w(uvw-uu'^2-vv'^2-ww'^2+2u'v'w')>0\end{array}\right\}\quad(24)$$

But (19) require also, in order that the conditions may be all satisfied, that

$$\left.\begin{array}{l}u'^2-vw<0\\v'^2-wu<0\\w'^2-uv<0\end{array}\right\}\quad(25)$$

hence replacing u, v, w, u', v', w', by their values in terms of the differential coefficients of Ω, and adopting the notation of determinants, the conditions finally become,

In the case of two variables

$$\begin{vmatrix} \frac{d^2\Omega}{dx^2}, & \frac{d^2\Omega}{dxdy} \\ \frac{d^2\Omega}{dydx}, & \frac{d^2\Omega}{dy^2} \end{vmatrix} > 0 \qquad (26)$$

In the case of three variables (x, y, z) the conditions will be

$$\begin{vmatrix} \frac{d^2\Omega}{dy^2}, & \frac{d^2\Omega}{dydz} \\ \frac{d^2\Omega}{dzdy}, & \frac{d^2\Omega}{dz^2} \end{vmatrix} > 0 \qquad \begin{vmatrix} \frac{d^2\Omega}{dz^2}, & \frac{d^2\Omega}{dzdx} \\ \frac{d^2\Omega}{dxdz}, & \frac{d^2\Omega}{dx^2} \end{vmatrix} > 0 \qquad \begin{vmatrix} \frac{d^2\Omega}{dx^2}, & \frac{d^2\Omega}{dxdy} \\ \frac{d^2\Omega}{dydx}, & \frac{d^2\Omega}{dy^2} \end{vmatrix} > 0$$

and

$$\frac{d^2\Omega}{dx^2} \begin{vmatrix} \frac{d^2\Omega}{dx^2}, & \frac{d^2\Omega}{dxdy}, & \frac{d^2\Omega}{dxdz} \\ \frac{d^2\Omega}{dydx}, & \frac{d^2\Omega}{dy^2}, & \frac{d^2\Omega}{dydz} \\ \frac{d^2\Omega}{dzdx}, & \frac{d^2\Omega}{dzdy}, & \frac{d^2\Omega}{dz^2} \end{vmatrix} > 0$$

$$\frac{d^2\Omega}{dy^2} \begin{vmatrix} \frac{d^2\Omega}{dx^2}, & \frac{d^2\Omega}{dxdy}, & \frac{d^2\Omega}{dxdz} \\ \frac{d^2\Omega}{dydx}, & \frac{d^2\Omega}{dy^2}, & \frac{d^2\Omega}{dydz} \\ \frac{d^2\Omega}{dzdx}, & \frac{d^2\Omega}{dzdy}, & \frac{d^2\Omega}{dz^2} \end{vmatrix} > 0$$

$$\frac{d^2\Omega}{dz^2} \begin{vmatrix} \frac{d^2\Omega}{dx^2}, & \frac{d^2\Omega}{dxdy}, & \frac{d^2\Omega}{dxdz} \\ \frac{d^2\Omega}{dydx}, & \frac{d^2\Omega}{dy^2}, & \frac{d^2\Omega}{dydz} \\ \frac{d^2\Omega}{dzdx}, & \frac{d^2\Omega}{dzdy}, & \frac{d^2\Omega}{dz^2} \end{vmatrix} > 0$$

Hence in general the final condition of a maximum or minimum will be one of the following

$$(\alpha) \qquad \mathfrak{A}\Theta > 0, \quad \mathfrak{B}\Theta > 0, \quad \mathfrak{C}\Theta > 0 \tag{27}$$

where

$$\Theta = \begin{vmatrix} \mathfrak{A}, & \mathfrak{F}, & \mathfrak{E} \\ \mathfrak{F}, & \mathfrak{B}, & \mathfrak{D} \\ \mathfrak{E}, & \mathfrak{D}, & \mathfrak{C} \end{vmatrix} \tag{28}$$

in which expressions $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}, \mathfrak{F}$ are of the form either

$$\mathrm{A}\Theta', \ \mathrm{B}\Theta', \ \mathrm{C}\Theta' \tag{29}$$

where

$$\Theta' = \begin{vmatrix} \mathrm{A}, & \mathrm{F}, & \mathrm{E} \\ \mathrm{F}, & \mathrm{B}, & \mathrm{D} \\ \mathrm{E}, & \mathrm{D}, & \mathrm{C} \end{vmatrix} \tag{30}$$

or

$$\Phi' = \begin{vmatrix} \mathrm{A}, & \mathrm{B} \\ \mathrm{B}, & \mathrm{C} \end{vmatrix} \tag{31}$$

and similarly A, B, C, D, E, F will be of the same form, and so on; or the conditions will take the form

$$(\beta) \qquad \Phi = \begin{vmatrix} \mathfrak{A}, & \mathfrak{B} \\ \mathfrak{B}, & \mathfrak{C} \end{vmatrix} > 0 \tag{32}$$

where $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ are of the form (29).

It is easily seen that, as the order of the final condition doubles itself every time the number of the variables is increased by unity, the order in the case of m variables will be 2^{m-1}.

If all the differentials of the function Ω, up to the $(2n+1)$th inclusively, vanish, it was seen above that

$$\mathrm{D}^{2(n+1)}\Omega$$

must always retain the same sign if there exists a maximum or minimum value of Ω. In order to find the conditions relative to this case a similar course must be pursued; thus writing for convenience

$$\mathrm{D}^{2n}\Omega = \Omega_{2n} \tag{33}$$

it will be necessary to find first the conditions that

$$\mathrm{D}^2\Omega_{2n}$$

may remain always of the same sign; this may be done by the rules given above. The order of the resulting condition will be as was seen above of the order

$$2^{m-1}$$

where m is the number of the variables; but as Ω_{2n} is of the order $2n$ with respect to the differentials, the order of the resulting condition will be

$$2^m n \tag{34}$$

with respect to the differentials. This must be looked upon as a quadratic, and the successive conditions deduced by degrees; thus the condition that

$$ax^p + bx^{p-1} + \ldots + ix^2 + jx + k$$
$$= (ax^{p-2} + bx^{p-3} + \ldots + i)x^2 + jx + k$$

may not change sign is

$$j^2 - 4k(ax^{p-2} + bx^{p-3} + \ldots + i) < 0$$

which again may be written

$$4k(ax^{p-4} + bx^{p-5} + \ldots)\, x^2 + 4klx - j^2 > 0$$

and so on.

On certain Formulæ made use of in Physical Astronomy.

Most of the following formulæ are given in the Cambridge Mathematical Journal, vol. IV., but the demonstrations are different.

By the principles of dynamics the equations of motion of a planet or satellite, considered as a material particle, are

$$\frac{d^2x}{dt^2} = X \quad \frac{d^2y}{dt^2} = Y \quad \frac{d^2z}{dt^2} = Z \tag{1}$$

where X Y Z are the resolved parts of the accelerating force parallel to the three axes of co-ordinates respectively. We shall find it convenient to use polar co-ordinates in future approximations; we will therefore transform (1). Let then

P = resolved part of the accelerating force parallel to the radius vector.

Q = resolved part of the perpendicular force in the plane of the orbit.

S = resolved part of the perpendicular force perpendicular to the plane of the orbit.

Also let

$$x = lr, \quad y = mr, \quad z = nr \tag{2}$$

whence, by differentiation,

$$\frac{dx}{dt} = l\frac{dr}{dt} + r\frac{dl}{dt}, \quad \frac{dy}{dt} = m\frac{dr}{dt} + r\frac{dm}{dt}, \quad \frac{dz}{dt} = n\frac{dr}{dt} + r\frac{dn}{dt} \tag{3}$$

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= l\frac{d^2r}{dt^2} + 2\frac{dl}{dt}\frac{dr}{dt} + r\frac{d^2l}{dt^2} \\ \frac{d^2y}{dt^2} &= m\frac{d^2r}{dt^2} + 2\frac{dm}{dt}\frac{dr}{dt} + r\frac{d^2m}{dt^2} \\ \frac{d^2z}{dt^2} &= n\frac{d^2r}{dt^2} + 2\frac{dn}{dt}\frac{dr}{dt} + r\frac{d^2n}{dt^2} \end{aligned}\right\} \tag{4}$$

Now

$$\left.\begin{aligned} P = lX + mY + nZ &= \frac{d^2r}{dt^2} + r\left\{l\frac{d^2l}{dt^2} + m\frac{d^2m}{dt^2} + n\frac{a^2n}{dt^2}\right\} \\ &= \frac{d^2r}{dt^2} - r\left\{\left(\frac{dl}{dt}\right)^2 + \left(\frac{dm}{dt}\right)^2 + \left(\frac{dn^2}{dt}\right)\right\} \end{aligned}\right\} \tag{5}$$

as is found by differentiating the equation

$$l^2 + m^2 + n^2 = 1$$

now from (3)

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = \left(\frac{ds}{dt}\right)^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left\{\left(\frac{dl}{dt}\right)^2 + \left(\frac{dm}{dt}\right)^2 + \left(\frac{dn}{dt}\right)^2\right\} \quad (7)$$

But if Θ be the angle through which the radius vector moves, we have also

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\Theta}{dt}\right)^2 \quad (8)$$

hence

$$\left(\frac{dl}{dt}\right)^2 + \left(\frac{dm}{dt}\right)^2 + \left(\frac{dn}{dt}\right)^2 = \left(\frac{d\Theta}{dt}\right)^2 = \omega^2 \quad (9)$$

where ω represents the angular velocity of the radius vector.

But if θ be the angle through which the radius vector moves, measured along the orbit, and Ω that through which it moves, measured perpendicular to the plane of the orbit,

$$\left(\frac{d\Theta}{dt}\right)^2 = \left(\frac{d\theta}{dt}\right)^2 + \left(\frac{d\Omega}{dt}\right)^2 \quad (10)$$

also

$$\left.\begin{aligned} \frac{dl}{dt} &= \frac{dl}{d\theta}\frac{d\theta}{dt} + \frac{dl}{d\Omega}\frac{d\Omega}{dt} \\ \frac{dm}{dt} &= \frac{dm}{d\theta}\frac{d\theta}{dt} + \frac{dm}{d\Omega}\frac{d\Omega}{dt} \\ \frac{dn}{dt} &= \frac{dn}{d\theta}\frac{d\theta}{dt} + \frac{dn}{d\Omega}\frac{d\Omega}{dt} \end{aligned}\right\} \quad (11)$$

so that

$$\left(\frac{dl}{d\theta}\right)^2 + \left(\frac{dm}{d\theta}\right)^2 + \left(\frac{dn}{d\theta}\right)^2 = 1 \quad (12)$$

$$\left(\frac{dl}{d\Omega}\right)^2 + \left(\frac{dm}{d\Omega}\right)^2 + \left(\frac{dn}{d\Omega}\right)^2 = 1 \quad (13)$$

and consequently also

$$\frac{dl}{d\theta}\frac{dl}{d\Omega} + \frac{dm}{d\theta}\frac{dm}{d\Omega} + \frac{dn}{d\theta}\frac{dn}{d\Omega} = 0 \quad (14)$$

hence we see that

$$l, m, n; \quad \frac{dl}{d\theta}, \frac{dm}{d\theta}, \frac{dn}{d\theta}; \quad \frac{dl}{d\Omega}, \frac{dm}{d\Omega}, \frac{dn}{d\Omega}$$

form the direction-cosines of a rectangular system; viz. the radius vector, the perpendicular to the radius vector in the plane of the orbit, and the perpendicular to the same line perpendicular to the plane of the orbit.

Hence

$$P = \frac{d^2 r}{dt^2} - \omega^2 r \tag{15}$$

Again, if we write

$$\frac{dl}{d\Theta} = l' \quad \frac{dm}{d\Theta} = m' \quad \frac{dn}{d\Theta} = n' \tag{16}$$

$$Q = l'X + m'Y + n'Z = 2\omega\frac{dr}{dt} + r\frac{d\omega}{dt} = \frac{1}{r}\frac{d(\omega r^2)}{dt} \tag{17}$$

Similarly, if

$$\frac{dl}{d\Omega} = \lambda \quad \frac{dm}{d\Omega} = \mu \quad \frac{dn}{d\Omega} = \nu \tag{18}$$

be the direction-cosines of a perpendicular to the plane of the orbit, and if ϖ represent the angular velocity of the plane of the orbit round the line perpendicular to the radius vector,

$$\left(\frac{d\lambda}{dt}\right)^2 + \left(\frac{d\mu}{dt}\right)^2 + \left(\frac{d\nu}{dt}\right)^2 = \varpi^2 \tag{19}$$

hence

$$\left.\begin{aligned} S = \lambda X + \mu Y + \nu Z &= r\left\{\lambda\frac{d^2 l}{dt^2} + \mu\frac{d^2 m}{dt^2} + \nu\frac{d^2 n}{dt^2}\right\} \\ &= r\left\{\frac{d\lambda}{dt}\frac{dl}{dt} + \frac{d\mu}{dt}\frac{dm}{dt} + \frac{d\nu}{dt}\frac{dn}{dt}\right\}\end{aligned}\right\} \tag{20}$$

taking account of the motion of the plane only, as is allowable by a theorem given at the end of this paper.

Also since $\frac{dl}{dt}\,\frac{dm}{dt}\,\frac{dn}{dt}$ are the direction-cosines of a line perpendicular to the plane containing the lines ($lmn\ \lambda\mu\nu$), so also are $\frac{d\lambda}{dt}\,\frac{d\mu}{dt}\,\frac{d\nu}{dt}$ the direction-cosines of a line perpendicular to the same plane, and consequently parallel to the first; we shall for convenience measure the angles so that the cosine of the angle between these lines is negative, hence

$$S = r\,\omega\,\varpi \tag{21}$$

Again, to determine the motion of the plane of the orbit we have

$$\left.\begin{aligned} l\frac{d\lambda}{dt} + m\frac{d\mu}{dt} + n\frac{d\nu}{dt} = 0 \\ \lambda\frac{d\lambda}{dt} + \mu\frac{d\mu}{dt} + \nu\frac{d\nu}{dt} = 0 \end{aligned}\right\} \tag{22}$$

whence

$$\frac{\frac{d\lambda}{dt}}{m\nu - n\mu} = \frac{\frac{d\mu}{dt}}{n\lambda - \nu l} = \frac{\frac{d\nu}{dt}}{l\mu - m\lambda} = \text{ȣ} \tag{23}$$

Since if

$$\left.\begin{aligned} \lambda l + \mu m + \nu n &= 0 \\ \text{then} \quad (m\nu - n\mu)^2 + (n\lambda - l\nu)^2 + (l\mu - m\lambda)^2 &= 1 \end{aligned}\right\} \tag{24}$$

also writing

$$h = r^2 \sqrt{\left\{ \left(\frac{dl}{dt}\right)^2 + \left(\frac{dm}{dt}\right)^2 + \left(\frac{dn}{dt}\right)^2 \right\}} \tag{25}$$

the equations for determining the motion of the planet or satellite in its orbit are

$$\frac{d^2 r}{dt^2} - \frac{h^2}{r^3} = \mathrm{P}, \quad \frac{dh}{dt} = \mathrm{Q}r \qquad (26)\ (27)$$

and those for the motion of the plane of the orbit

$$\frac{\frac{d\lambda}{dt}}{m\nu - n\mu} = \frac{\frac{d\mu}{dt}}{n\lambda - l\nu} = \frac{\frac{d\nu}{dt}}{l\mu - m\lambda} = \frac{\mathrm{S}r}{h} \tag{28}$$

We will now calculate the expressions for P Q S.

Let $m\, m'\, m''$ be a free system of three mutually attracting bodies; we may suppose one of them to become fixed (suppose m'') if we apply to the system forces equal and opposite to those which act on m''. Let

$$r = (m'', m), \qquad r' = (m'', m'), \qquad \delta = (m, m'), \tag{29}$$

Then the forces which act on m are

$\frac{m''}{r^2}$ along mm'' $\qquad$ $\frac{m'}{\delta^2}$ along mm'

$\frac{m}{r^2}$ along mm'' $\qquad$ $-\frac{m'}{r'^2}$ parallel to $m''m'$

Then

$$\left.\begin{aligned} -\mathrm{X} &= \frac{m + m''}{r^3} x + \frac{m'}{r'^3} x' + \frac{m'}{\delta^3}(x - x') \\ -\mathrm{Y} &= \frac{m + m''}{r^3} y + \frac{m'}{r'^3} y' + \frac{m'}{\delta^3}(y - y') \\ -\mathrm{Z} &= \frac{m + m''}{r^3} z + \frac{m'}{r'^3} z' + \frac{m'}{\delta^3}(z - z') \end{aligned}\right\} \tag{30}$$

also

$$\mathrm{P} = l\mathrm{X} + m\mathrm{Y} + n\mathrm{Z}, \quad \mathrm{Q} = l'\mathrm{X} + m'\mathrm{Y} + n'\mathrm{Z}, \quad \mathrm{S} = \lambda\mathrm{X} + \mu\mathrm{Y} + \nu\mathrm{Z} \tag{31}$$

hence

$$\left.\begin{aligned}
-\mathrm{P} &= \frac{m+m''}{r^2} + m'\left(\frac{lx'+my'+nz'}{r'^2} + \frac{l(x-x')+m(y-y')+n(z-z')}{\delta^2}\right)\\
-\mathrm{Q} &= m'\left(\frac{l'x'+m'y'+n'z'}{r'^2} + \frac{l'(x-x')+m(y-y')+n(z-z')}{\delta^2}\right)\\
-\mathrm{S} &= m'\left(\frac{\lambda x'+\mu y'+\nu z'}{r'^2} + \frac{\lambda(x-x')+\mu(y-y')+\nu(z-z')}{\delta^2}\right)
\end{aligned}\right\} \tag{32}$$

or writing

$$-\mathrm{R} = m'\frac{xx'+yy'+zz'}{r'^3} - \frac{m'}{\delta} \tag{33}$$

$$\mathrm{P} = -\frac{m+m''}{r^2} + l\frac{d\mathrm{R}}{dx} + m\frac{d\mathrm{R}}{dy} + n\frac{d\mathrm{R}}{dz} \tag{34}$$

$$\mathrm{Q} = \frac{dl}{d\Theta}\frac{d\mathrm{R}}{dx} + \frac{dm}{d\Theta}\frac{d\mathrm{R}}{dy} + \frac{dn}{d\Theta}\frac{d\mathrm{R}}{dz} \tag{35}$$

$$\mathrm{S} = \lambda\frac{d\mathrm{R}}{dx} + \mu\frac{d\mathrm{R}}{dy} + \nu\frac{d\mathrm{R}}{dz} \tag{36}$$

Now

$$\left.\begin{aligned}
\frac{d\mathrm{R}}{dx} &= \frac{d\mathrm{R}}{dl}\frac{dl}{dx} + \frac{d\mathrm{R}}{dm}\frac{dm}{dx} + \frac{d\mathrm{R}}{dn}\frac{dn}{dx} + \frac{d\mathrm{R}}{dr}\,l\\
\frac{d\mathrm{R}}{dy} &= \frac{d\mathrm{R}}{dl}\frac{dl}{dy} + \frac{d\mathrm{R}}{dm}\frac{dm}{dy} + \frac{d\mathrm{R}}{dn}\frac{dn}{dy} + \frac{d\mathrm{R}}{dr}\,m\\
\frac{d\mathrm{R}}{dz} &= \frac{d\mathrm{R}}{dl}\frac{dl}{dz} + \frac{d\mathrm{R}}{dm}\frac{dm}{dz} + \frac{d\mathrm{R}}{dn}\frac{dn}{dz} + \frac{d\mathrm{R}}{dr}\,n
\end{aligned}\right\} \tag{37}$$

and

$$\left.\begin{aligned}
\frac{dl}{dx} &= \frac{m^2+n^2}{r} & \frac{dl}{dy} &= -\frac{lm}{r} & \frac{dl}{dz} &= -\frac{ln}{r}\\
\frac{dm}{dx} &= -\frac{ml}{r} & \frac{dm}{dy} &= \frac{n^2+l^2}{r} & \frac{dm}{dz} &= -\frac{mn}{r}\\
\frac{dn}{dx} &= -\frac{nl}{r} & \frac{dn}{dy} &= -\frac{nm}{r} & \frac{dn}{dz} &= \frac{l^2+m^2}{r}
\end{aligned}\right\} \tag{38}$$

hence

$$\left.\begin{aligned}
r\frac{d\mathrm{R}}{dx} &= \frac{d\mathrm{R}}{dl} - l\left(l\frac{d\mathrm{R}}{dl} + m\frac{d\mathrm{R}}{dm} + n\frac{d\mathrm{R}}{dn} - r\frac{d\mathrm{R}}{dr}\right)\\
r\frac{d\mathrm{R}}{dy} &= \frac{d\mathrm{R}}{dm} - m\left(l\frac{d\mathrm{R}}{dl} + m\frac{d\mathrm{R}}{dm} + n\frac{d\mathrm{R}}{dn} - r\frac{d\mathrm{R}}{dr}\right)\\
r\frac{d\mathrm{R}}{dz} &= \frac{d\mathrm{R}}{dn} - n\left(l\frac{d\mathrm{R}}{dl} + m\frac{d\mathrm{R}}{dm} + n\frac{d\mathrm{R}}{dn} - r\frac{d\mathrm{R}}{dr}\right)
\end{aligned}\right\} \tag{39}$$

hence

$$\mathrm{P} = \frac{m+m''}{r^2} + l\frac{d\mathrm{R}}{dx} + m\frac{d\mathrm{R}}{dy} + n\frac{d\mathrm{R}}{dz} = \frac{m+m''}{r^2} + \frac{d\mathrm{R}}{dr} \tag{40}$$

also

$$\left.\begin{aligned} Q &= \frac{dl}{d\Theta}\frac{dR}{dx} + \frac{dm}{d\Theta}\frac{dR}{dy} + \frac{dn}{d\Theta}\frac{dR}{dz} \\ &= \frac{1}{r}\left(\frac{dR}{dl}\frac{dl}{d\Theta} + \frac{dR}{dm}\frac{dm}{d\Theta} + \frac{dR}{dn}\frac{dn}{d\Theta}\right) \\ &= \frac{1}{r}\frac{dR}{d\Theta} \end{aligned}\right\} \tag{41}$$

similarly

$$\left.\begin{aligned} S &= \frac{dl}{d\Omega}\frac{dR}{dx} + \frac{dm}{d\Omega}\frac{dR}{dy} + \frac{dn}{d\Omega}\frac{dR}{dz} \\ &= \frac{1}{r}\left(\frac{dR}{dl}\frac{dl}{d\Omega} + \frac{dR}{dm}\frac{dm}{d\Omega} + \frac{dR}{dn}\frac{dn}{d\Omega}\right) \\ &= \frac{1}{r}\frac{dR}{d\Omega} \end{aligned}\right\} \tag{42}$$

where $d\Omega$ is the small angle through which the plane of the orbit turns round a line perpendicular to the radius vector. If we wish to express this in terms of the partial differential co-efficient of R with respect to the inclination (i) of the plane of the orbit to that of the ecliptic, it is easily shown that if ϑ be the angle between the radius vector and the line of nodes we have merely to write

$$d\Omega = \sin\vartheta\, dr$$

for if

$$\frac{di}{dt} = \text{variation of inclination}$$

and if α, β, γ, be the direction-cosines of the line of nodes, the components of $\frac{di}{dt}$ will be

$$\alpha\frac{di}{dt},\quad \beta\frac{di}{dt},\quad \gamma\frac{di}{dt},$$

and the resolved part in the direction of the line round which $\frac{d\Omega}{dt}$ is measured will be

$$\left(\alpha\frac{dl}{d\Omega} + \beta\frac{dm}{d\Omega} + \gamma\frac{dn}{d\Omega}\right)\frac{di}{dt}$$
$$= \sin\vartheta\frac{di}{dt}$$

where ϑ is the angle between the radius vector and the line of nodes; so that the expression for S becomes also

$$S = \frac{1}{r\sin\vartheta}\frac{dR}{di} \tag{43}$$

also if T represent the force along the tangent

$$T = X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds} = -\frac{m+m''}{r^2}\frac{dr}{ds} + \frac{dR}{ds} = \frac{dv}{dt} \tag{44}$$

and the results finally give

$$\frac{d^2r}{dt^2} - \frac{h^2}{r^3} + \frac{m''+m}{r^2} = \frac{dR}{ds}, \quad \frac{dh}{dt} = 2\frac{dR}{d\Theta} \qquad (45)\,(46)$$

$$\frac{\frac{d\lambda}{dt}}{m\nu - n\mu} = \frac{\frac{d\mu}{dt}}{n\lambda - l\nu} = \frac{\frac{d\nu}{dt}}{l\mu - m\lambda} = \frac{1}{h}\frac{dR}{d\Omega} \tag{47}$$

$$v^2 = \frac{2(m+m'')}{r} + 2\int\frac{dR}{dt}\,dt \tag{48}$$

We will now integrate the equations of motion of an undisturbed planet or satellite. In this case we have

$$m' = 0 \qquad R = 0 \tag{49}$$

and consequently

$$\frac{d\lambda}{dt} = 0 \qquad \frac{d\mu}{dt} = 0 \qquad \frac{d\nu}{dt} = 0 \qquad \frac{dh}{dt} = 0 \tag{50}$$

whence by integration

$$\lambda = \text{const.} \qquad \mu = \text{const.} \qquad \nu = \text{const.} \qquad h = \text{const.} \tag{51}$$

hence the plane of the orbit is fixed. Let θ be the value of Θ in this case, and let

$$m + m'' = \kappa \tag{52}$$

and in this case (21) and (40) become

$$r^2\frac{d\theta}{dt} = h \qquad v^2 = \frac{2\kappa}{r} + C \tag{53}$$

also writing for convenience

$$r = \frac{1}{u} \tag{54}$$

the second of (48) becomes by means of the first

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2\kappa}{h^2}u + \frac{C}{h^2} \tag{55}$$

which by integration gives

$$u = \frac{\kappa}{h^2} + \frac{1}{h}\sqrt{Ch^2 + \kappa^2}\,.\cos(\theta - \varpi) \tag{56}$$

which is the equation to a conic section, consequently a planet or satellite if undisturbed would move in a conic section. Now since the planets and satellites move approximately in ellipses, we will determine the elements of the ellipse represented by (56) in terms of the constants introduced by the integrations. Comparing the above equation with the usual equation to an ellipse

$$u = \frac{1}{a(1-e^2)} + \frac{e}{a(1-e^2)} \cos(\theta - \pi) \tag{57}$$

we have

$$h^2 = \kappa a (1-e^2) \qquad C = -\frac{\kappa}{a} \tag{58}$$

conversely we may determine these quantities in terms of the initial circumstances of motion, thus :

Let V be the initial velocity of the planet or satellite,

ρ - - - distance - - -

γ - - angle between the directions of ρ and V ;

then

$$\left.\begin{aligned} h = \rho V \sin\gamma \quad C = V^2 - \frac{2\kappa}{\rho} \quad a = \frac{\kappa}{2\frac{\kappa}{\rho} - V^2} \\ e^2 = 1 - \left(\frac{\rho V \sin\gamma}{\kappa}\right)^2 \left(\frac{2\kappa}{\rho} - V^2\right) \end{aligned}\right\} \tag{59}$$

It remains for us to determine the time of the body describing a given arc ; this may thus be found. We will have from (53) and (58)

$$\left(\frac{ds}{dt}\right)^2 = r^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dr}{dt}\right)^2 = \frac{\kappa a(1-e^2)}{r^2} + \left(\frac{dr}{dt}\right)^2 = \kappa\left(\frac{2}{r} - \frac{1}{a}\right) \tag{60}$$

whence

$$\frac{dr}{dt} = \sqrt{\frac{\kappa}{a}} \frac{\sqrt{a^2 e^2 - (a-r)^2}}{r} \tag{61}$$

whence integrating and supposing θ measured from perihelion

$$t = \sqrt{\frac{a}{\kappa}} \left\{ a \cos^{-1} \frac{a-r}{ae} - \sqrt{a^2e^2 - (a-r)^2} \right\} \tag{62}$$

at aphelion $r = a(1+e)$, and $t = \frac{1}{2}$ periodic time $= \frac{T}{2}$

$$\therefore T = \frac{2\pi a^{\frac{3}{2}}}{\kappa^{\frac{1}{2}}} \tag{63}$$

We will now introduce some auxiliary quantities which are found useful in Astronomy. In the ellipse

$$r = \frac{a\,(1-e^2)}{1 + e\cos\theta} \tag{64}$$

θ is called the true anomaly; let us assume

$$u = \cos^{-1}\frac{a-r}{ae}, \text{ or } r = a\,(1-e\cos u) \tag{65}$$

(this u is of course different from the u employed above in (54), &c.) whence

$$(1 - e\cos u)\,(1 + e\cos\theta) = 1 - e^2$$

or

$$\cos\theta = \frac{\cos u - e}{1 - e\cos u} \text{ or } \frac{1-\cos\theta}{1+\cos\theta} = \frac{1+e}{1-e}\,\frac{1-\cos u}{1+\cos u}$$

whence

$$\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\,\tan\frac{u}{2} \tag{66}$$

u is called the *eccentric anomaly;* it will be necessary to determine its geometrical value. If the ellipse change form so that it becomes a circle, then

$$e = 0,\ \theta = u,$$

that is θ and u coincide; but if

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

be the equation to an ellipse, the equation to the circle described on its major axis will be

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$$

in other words, it is possible to pass from an ellipse to a circle described on the major axis of the ellipse by increasing the ordinates in the ratio $a : b$; or vice versâ by decreasing the ordinates in the ratio $b : a$. But besides this it must be remembered that when $e = 0$ the foci of the ellipse fall into the common centre of the two figures. These considerations will enable us to pass at once from θ to u; let C be the common centre of the ellipse and circle, S the focus of the ellipse at which the origin of co-ordinates is taken; then the rectangular co-ordinates of any point P on the ellipse from the origin S will be S N, N P (suppose), and the radius vector S P and θ will be the angle N S P or its supplement;

if then the ordinate N P be increased in the ratio $a : b$ and become N P′, P′ will be the point in the circle corresponding to the point P in the ellipse; but as the origin is now removed from S to C (as was remarked above) the co-ordinates of P′ will be C N, N P′, and the radius vector C P′ and u will be the angle N C P′ or its supplement. Thus the geometrical meaning of the eccentric anomaly is completely determined.

$$dt = \sqrt{\frac{a}{\kappa}} \frac{rdr}{\sqrt{a^2e^2-(a-r)^2}} = \sqrt{\frac{a}{\kappa}} \frac{rdr}{ac \sin u} = \frac{a^{\frac{3}{2}}}{\sqrt{\kappa}} (1-e \cos u)\, du \qquad (67)$$

and supposing the time and θ measured from perihelion

$$\frac{\sqrt{\kappa}}{a^{\frac{3}{2}}} t = u - e \sin u = nt \text{ suppose} \qquad (68)$$

hence $n = \frac{2u}{\mathrm{T}}$ = average angular velocity

and nt = mean anomaly.

We will now give some expansions of these quantities in terms of one another, deducing the actual series as far as a few terms only.

(1.) To expand the radius vector in terms of the true anomaly.

$$\begin{aligned} r &= \frac{a(1-e^2)}{1+e \cos \theta} \\ &= a(1-e^2)(1-e \cos \theta + e^2 \cos^2\theta - e^3 \cos^3\theta) \\ &= a(1-e \cos \theta + e^2 \cos^2\theta - e^3 \cos^3\theta - e^2 + e^3 \cos \theta) \\ &= a(1-e \cos\theta - \frac{e^2}{2} - \frac{e^2}{2} + \frac{e^2}{2}\cos^2\theta + \frac{e^2}{2}\cos^2\theta - \frac{4e^3}{4}\cos^3\theta + \frac{e^3}{4}\cos\theta + \frac{3e^3}{4}\cos\theta) \\ &= a(1-\frac{e^2}{2} - \left(e - \frac{e^3}{4}\right)\cos\theta + \frac{e^2}{2}\cos 2\theta - \frac{e^3}{4}\cos 3\theta) \qquad (69) \end{aligned}$$

(2.) To expand the eccentric anomaly in terms of the mean.

$$\begin{aligned} u &= nt + e \sin u \\ &= nt + \frac{e}{1}\sin nt + \frac{e^2}{1.2}\frac{d}{dnt}(\sin^2 nt) + \frac{e^2}{1.2.3}\frac{d^2}{dnt^2}(\sin^3 nt) \text{ By Lagrange's theorem.} \\ &= nt + \frac{e}{1}\sin nt + \frac{e^2}{1.2}\sin 2nt + \frac{e^3}{1.2.3}(6 \sin nt \cos^2 nt - 3 \sin^3 nt) \\ &= nt + \frac{e}{1}\sin nt + \frac{e^2}{1.2}\sin 2nt + \frac{e^3}{1.2.3}(6 \sin nt - 9 \sin^3 nt) \\ &= nt + \frac{e}{1}\sin nt + \frac{e^2}{1.2}\sin 2nt + \frac{e^3}{1.2.3}\left(\frac{9}{4}\sin 3nt - \frac{3}{4}\sin nt\right) \\ &= nt + \left(\frac{e}{1} + \frac{e^3}{8}\right)\sin nt + \frac{e^2}{1.2}\sin 2nt + \frac{3e^3}{8}\sin 3nt \qquad (70) \end{aligned}$$

Again, by Lagrange's theorem,

$$\operatorname{Cos} u = \cos nt + e \sin nt \frac{d}{dnt}(\cos nt) + \frac{e^2}{1.2}\frac{d}{dnt}\left\{\sin^2 nt \frac{d}{dnt}(\cos nt)\right\}$$

$$= \cos nt - e \sin^2 nt - \frac{e^2}{1.2}\frac{d}{dnt}(\sin^3 nt)$$

$$= \cos nt - e\frac{1-\cos 2nt}{2} + \frac{e^2}{1.2}\left\{\frac{3}{4}\cos 3nt - \frac{3}{4}\cos nt\right\}$$

$$= -\frac{e}{2} + \left(1 - \frac{3}{8}e^2\right)\cos nt + \frac{e}{2}\cos 2nt + \frac{3}{8}e^2 \cos 3nt \qquad (71)$$

(3.) To expand the radius vector in terms of the mean anomaly.

$$\frac{r}{a} = 1 - e\cos u$$

$$= 1 - e\cos nt + \frac{e^2}{2}(1 - \cos 2nt) - \frac{e^3}{8}(3\cos 3nt - 3\cos nt)$$

$$= 1 + \frac{e^2}{2} - \left(e - \frac{3e^3}{8}\right)\cos nt - \frac{e^2}{2}\cos 2nt - \frac{3e^3}{8}\cos 3nt \qquad (72)$$

(4.) To expand the mean anomaly in terms of the true.

$$dt = \frac{r^2 d\theta}{h} = \frac{r^2 d\theta}{\sqrt{\kappa a(1-e^2)}}$$

$$ndt = \frac{r^2 d\theta}{a^2\sqrt{1-e^2}} = (1-e^2)^{\frac{3}{2}}\frac{d\theta}{(1+e\cos\theta)^2}$$

$$= \left(1 - \frac{3}{2}e^2\right)(1 - 2e\cos\theta + 3e^2\cos^2\theta - 4e^3\cos^3\theta)$$

$$= 1 - 2e\cos\theta + 3e^2\cos^2\theta - 4e^3\cos^3\theta - \frac{3}{2}e^2 + 3e^3\cos\theta$$

$$= 1 - 2e\cos\theta + \frac{3}{2}e^2\cos 2\theta - e^3\cos 3\theta$$

$$\therefore\ nt = \theta - 2e\sin\theta + \frac{3}{4}e^2\sin 2\theta - \frac{e^3}{3}\sin 3\theta \qquad (73)$$

To expand the true anomaly in terms of the mean. We have above

$$\theta = nt + e\left(2\sin\theta - \frac{3}{4}e\sin 2\theta - \frac{e^2}{3}\sin 3\theta\right)$$

$$= nt + e\phi(\theta)$$

$$= nt + e\phi(nt) + \frac{e^2}{1.2}\frac{d}{dnt}\{\phi(nt)\}^2 + \frac{e^3}{1.2.3}\frac{d^2}{dnt^2}\{\phi(nt)\}^3$$

$$= nt + \left(2e - \frac{e^3}{4}\right)\sin nt + \frac{5}{4}e^2\sin 2nt + \frac{13}{12}e^3\sin 3nt \qquad (74)$$

To expand the true anomaly in terms of the eccentric.

In equation (66), writing for convenience $\frac{1+e}{1-e} = k^2$, and putting the whole in an exponential form

$$\frac{\varepsilon^{2\theta\sqrt{-1}}-1}{\varepsilon^{2\theta\sqrt{-1}}+1} = k\frac{\varepsilon^{2u\sqrt{-1}}-1}{\varepsilon^{2u\sqrt{-1}}+1}$$

whence

$$\varepsilon^{2x\sqrt{-1}} = \frac{(k+1)\varepsilon^{2u\sqrt{-1}}-(k-1)}{(k+1)-(k-1)\varepsilon^{2u\sqrt{-1}}} = \frac{\varepsilon^{2u\sqrt{-1}}-\lambda}{1-\lambda\varepsilon^{2u\sqrt{1-}}}$$

let

$$\varepsilon^{2u\sqrt{-1}} = z \qquad \therefore \quad \varepsilon^{2\theta\sqrt{-1}} = \frac{z-\lambda}{1-\lambda z}$$

whence taking logarithms

$$2\theta\sqrt{-1} = \log z\frac{1-\frac{\lambda}{z}}{1-\lambda z}$$

$$= \log z - \frac{\lambda}{z} - \frac{\lambda^2}{z^2} - \frac{\lambda^3}{z^3} - \ldots + \lambda z + \lambda^2 z^2 + \lambda^3 z^3 + \ldots$$

$$= 2u\sqrt{-1} + 2\sqrt{-1}(\lambda \sin 2u + \frac{\lambda^2}{2}\sin 4u + \ldots)$$

$$\therefore \quad \theta = u + 2\lambda\sin u + 2\frac{\lambda^2}{2}\sin 2u + 2\frac{\lambda^3}{3}\sin 3u \tag{75}$$

$$\text{where} \quad \lambda = \frac{e}{1-\sqrt{1-e^2}}$$

The problem of disturbed motion will be much simplified by the following theorem.

The differential equations of the motion m expressed in rectangular co-ordinates may be written thus,

$$\left.\begin{aligned} \frac{d^2x}{dt^2} + \frac{\mu x}{r^3} &= \frac{d\mathrm{R}}{dx} \\ \frac{d^2y}{dt^2} + \frac{\mu y}{r^3} &= \frac{d\mathrm{R}}{dy} \\ \frac{d^2z}{dt^2} + \frac{\mu z}{r^3} &= \frac{d\mathrm{R}}{dz} \end{aligned}\right\} \tag{76}$$

or as they may be written

$$\frac{\frac{d^2x}{dt^2} - \frac{d\mathrm{R}}{dx}}{x} = \frac{\frac{d^2y}{dt^2} - \frac{d\mathrm{R}}{dy}}{y} = \frac{\frac{d^2z}{dt^2} - \frac{d\mathrm{R}}{dz}}{z} = -\frac{\mu}{r^3} \tag{77}$$

if the co-ordinates be changed by the formulæ

$$\xi = ax + by + cz, \quad \eta = a'x + b'y + c'z, \quad \zeta = a''x + b''y + c''z \tag{78}$$

in which the nine direction-cosines are subject to the six equations of condition

$$\left.\begin{array}{ll} a^2 + b^2 + c^2 = 1 & a'a'' + b'b'' + c'c'' = 0 \\ a'^2 + b'^2 + c'^2 = 1 & a''a + b''b + c''c = 0 \\ a''^2 + b''^2 + c''^2 = 1 & aa' + bb' + cc' = 0 \end{array}\right\} \tag{79}$$

whence also

$$x^2 + y^2 + z^2 = \xi^2 + \eta^2 + \zeta^2 \tag{80}$$

then using the factors a, b, c; a', b', c'; a'', b'', c'', respectively, in the equation (77), there results the following system,

$$\left.\begin{array}{l} \dfrac{\dfrac{d^2\xi}{dt^2} - a\dfrac{d\mathrm{R}}{dx} - b\dfrac{d\mathrm{R}}{dy} - c\dfrac{d\mathrm{R}}{dz}}{\xi} \\ = \dfrac{\dfrac{d^2y}{dt^2} - a'\dfrac{d\mathrm{R}}{dx} - b'\dfrac{d\mathrm{R}}{dy} - c'\dfrac{d\mathrm{R}}{dz}}{\eta} \\ = \dfrac{\dfrac{d^2\zeta}{dt^2} - a''\dfrac{d\mathrm{R}}{dx} - b''\dfrac{d\mathrm{R}}{dy} - c''\dfrac{d\mathrm{R}}{dz}}{\zeta} = -\dfrac{\mu}{\rho^3} \end{array}\right\} \tag{81}$$

where

$$\rho^2 = \xi^2 + \eta^2 + \zeta^2 \tag{82}$$

But

$$\left.\begin{array}{l} \dfrac{d\mathrm{R}}{d\xi} = \dfrac{d\mathrm{R}}{dx}\dfrac{dx}{d\xi} + \dfrac{d\mathrm{R}}{dy}\dfrac{dy}{d\xi} + \dfrac{d\mathrm{R}}{dz}\dfrac{dz}{d\xi} \\ \qquad = a\dfrac{d\mathrm{R}}{dx} + b\dfrac{d\mathrm{R}}{dy} + c\dfrac{d\mathrm{R}}{dz} \\ \dfrac{d\mathrm{R}}{d\eta} = \dfrac{d\mathrm{R}}{dx}\dfrac{dx}{d\eta} + \dfrac{d\mathrm{R}}{dy}\dfrac{dy}{d\eta} + \dfrac{d\mathrm{R}}{dz}\dfrac{dz}{d\eta} \\ \qquad = a'\dfrac{d\mathrm{R}}{dx} + b'\dfrac{d\mathrm{R}}{dy} + c'\dfrac{d\mathrm{R}}{dz} \\ \dfrac{d\mathrm{R}}{d\zeta} = \dfrac{d\mathrm{R}}{dx}\dfrac{dx}{d\zeta} + \dfrac{d\mathrm{R}}{dy}\dfrac{dy}{d\zeta} + \dfrac{d\mathrm{R}}{dz}\dfrac{dz}{d\zeta} \\ \qquad = a''\dfrac{d\mathrm{R}}{dx} + b''\dfrac{d\mathrm{R}}{dy} + c''\dfrac{d\mathrm{R}}{dz} \end{array}\right\} \tag{83}$$

hence (81) becomes

$$\frac{\dfrac{d^2\xi}{dt^2} - \dfrac{d\mathrm{R}}{d\xi}}{\xi} = \frac{\dfrac{d^2\eta}{dt^2} - \dfrac{d\mathrm{R}}{d\eta}}{\eta} = \frac{\dfrac{d^2\zeta}{dt^2} - \dfrac{d\mathrm{R}}{d\zeta}}{\zeta} = -\frac{\mu}{\rho^3} \tag{84}$$

But since three of the nine direction-cosines are entirely arbitrary, they may always be chosen so as to destroy the third of the expressions in (84), so that there will be only two equations of motion remaining. Hence if the moon be first supposed to move in the plane of the ecliptic the results so obtained will be true, when the inclination is taken into account.

On the Calculus of Variations.

I.—FUNDAMENTAL FORMULÆ.

By means of the Calculus of Differentials the properties of given functions of any variables may be investigated, and the changes of the former traced through values of the latter differing from one another by insensible quantities; or, conversely, the values or a system of values of the variables, where certain properties of the functions (represented by given combinations of their differential coefficients, or of the differentials of the variables) have any proposed values, may be found. There exists, however, a more extensive class of problems, in which the forms of the functions are themselves unknown, and consequently become the subject of investigation. The method by which problems of this kind are solved is called the Calculus of Variations.

This Calculus is analogous to that of Differentials; the operations of the one are in form exactly similar to those of the other, so that certain corresponding points in the two may be advantageously compared in the course of the present investigations. In order to understand this more clearly, consider the function

$$\Omega\,(u, v, w, \ldots) = 0, \tag{1}$$

in which expression

$$u = f_1\,(x, y, z, \ldots),\ v = f_2\,(x, y, z, \ldots),\ w = f_3\,(x, y, z, \ldots)\ \ldots \tag{2}$$

the symbols $\Omega, f_1, f_2, f_3, \ldots$ representing any functions whatever. Now the functions treated of in the Calculus of Differentials (or, that which is nearly equivalent, the Differential Calculus) are such that their forms are always given, and consequently in this case

$$u = a,\ v = b,\ w = c,\ \ldots . \tag{3}$$

where $a, b, c, \ldots$ are constants made variable only in certain classes of problems, such as envelopes, variation of elements of motion, &c.; the object of the former class (which is most to the present purpose) being to eliminate $x, y, z, \ldots$ by means of the equations (3) and their differentials, the original variables $x, y, z, \ldots$ being considered constant, and the original constants $a, b, c, \ldots$ variable in the differentiations.

But even in this case the form of the functions represented by (2) is known, and it is possible to pass only from one individual to another of the same class, and not from one class to another: Thus, for example, in analytical geometry any surface under consideration is in the case of envelopes, or ultimate intersections, supposed to change form according to a certain law, the class to which it belongs remaining always the same. As however in the Calculus of Variations the forms of the functions must, when determined, hold good generally, *i. e.* for all values of the variables $x, y, z, \ldots$, it is clear that the changes of the two sets of variables $x, y, z, \ldots u, v, w, \ldots$ must be simultaneously taken into account. In this case it will be convenient to introduce the new symbol δ, and to represent the changes or (as they will be hereafter called) the *variations* of $x, y, z, \ldots u, v, w, \ldots$ by

$$\left.\begin{array}{l}\delta x,\ \delta y,\ \delta z,\ \ldots\\ \delta u,\ \delta v,\ \delta w,\ \ldots\end{array}\right\} \tag{4}$$

in the same way as their *differentials* were represented by

$$\left.\begin{array}{l}dx,\ dy,\ dz,\ \ldots\\ du,\ dv,\ dw,\ \ldots\end{array}\right\} \tag{5}$$

and similarly their successive variations by

$$\left.\begin{array}{l}\delta^n x,\ \delta^n y,\ \delta^n z,\ \ldots\\ \delta^n u,\ \delta^n v,\ \delta^n w,\ \ldots\end{array}\right\} \tag{6}$$

in the same way as their successive differentials were represented by

$$\left.\begin{array}{l}d^n x,\ d^n y,\ d^n z,\ \ldots\\ d^n u,\ d^n v,\ d^n w,\ \ldots\end{array}\right\} \tag{7}$$

So also proceeding according to the rules above given, and representing the partial variations of any function Ω by the symbols

$$\delta_x\Omega,\ \delta_y\Omega,\ \delta_z\Omega,\ \ldots \tag{8}$$

in the same way as the partial differentials are represented by the symbols

$$d_x\Omega,\ d_y\Omega,\ d_z\Omega,\ \ldots \tag{9}$$

we shall have as the expression for the total variation of the same function

$$\delta\Omega = \frac{\delta_x\Omega}{\delta x}\delta x + \frac{\delta_y\Omega}{\delta y}\delta y + \frac{\delta_z\Omega}{\delta z}\delta z + \ldots \tag{10}$$

a formula analogous to the differential

$$d\Omega = \frac{d_x\Omega}{dx}dx + \frac{d_y\Omega}{dy}dy + \frac{d_z\Omega}{dz}dz + \ \ldots \tag{11}$$

Before proceeding further it will be well to consider a particular case, in order to throw as much light as possible upon the nature of variations. Consider the case of a surface whose equation is

$$F(x, y, z) = 0 \tag{12}$$

Now by means of the Differential Calculus we are enabled to pass from one point of the surface to another, and also to discover certain geometrical properties which it possesses. For instance, the direction of the normal at any point may be determined by any two of the equations

$$\cos a = \frac{1}{Q}\frac{d_x F}{dx}, \ \cos \beta = \frac{1}{Q}\frac{d_y F}{dy}, \ \cos \gamma = \frac{1}{Q}\frac{d_z F}{dz} \tag{13}$$

where

$$Q^2 = \left(\frac{d_x F}{dx}\right)^2 + \left(\frac{d_y F}{dy}\right)^2 + \left(\frac{d_z F}{dz}\right)^2 \tag{14}$$

and by assuming arbitrarily values for two of the variables as

$$x = \mathrm{x}, \ y = \mathrm{y}, \ \text{or } z = \mathrm{z} \tag{15}$$

the remaining two can be determined by the equation

$$F(x, \mathrm{y}, \mathrm{z}) = 0, \ \text{or } F(\mathrm{x}, y, \mathrm{z}) = 0, \ \text{or } F(\mathrm{x}, \mathrm{y}, z) \tag{16}$$

Whence we can find the values of the angles α, β, γ by means of the equations (13), in which x, y, z are to be written for x, y, z after the differentiations have been performed. Conversely we may find a point where the normal has any given direction (provided that the surface admits of such), that is, where two of the direction-cosines have any arbitrary values, such as those given by any two of the equations

$$\cos a = l, \ \cos \beta = m, \ \cos \gamma = n \tag{17}$$

by means of the equation to the surface, and the corresponding two of the system

$$\frac{1}{Q}\frac{d_x F}{dx} = l, \ \frac{1}{Q}\frac{d_y F}{dy} = m, \ \frac{1}{Q}\frac{d_z F}{dz} = n \tag{18}$$

Suppose however that we have a more general case, and consider any equation of the form

$$F(u, v, w) = 0 \tag{19}$$

where

$$u = f_1(x, y, z), \ v = f_2(x, y, z), \ w = f_3(x, y, z) \tag{20}$$

Now if u, v, w be determinate functions of x, y, z, whose form does not change, the case will be precisely similar to the former one ; but if the forms of u, v, w vary, we shall still proceed in a similar way, which will give rise to the three expressions

$$\frac{\delta_x F}{\delta x}, \quad \frac{\delta_y F}{\delta y}, \quad \frac{\delta_z F}{\delta z}$$

or any other that may be required. Thus taking the case before considered, the equations (13), (14), and (18) would retain the same form as before; but, by the principles of differentiation,

$$\frac{\delta_x F}{\delta x} = \frac{\delta_u F}{\delta u}\frac{\delta_x u}{\delta x} + \frac{\delta_v F}{\delta v}\frac{\delta_x v}{\delta x} + \frac{\delta_w F}{\delta w}\frac{\delta\, w}{\delta x}$$

$$\frac{\delta_y F}{\delta y} = \frac{\delta_u F}{\delta u}\frac{\delta_y u}{\delta y} + \frac{\delta_v F}{\delta v}\frac{\delta_y v}{\delta y} + \frac{\delta_w F}{\delta w}\frac{\delta_y w}{\delta y}$$

$$\frac{\delta_z F}{\delta z} = \frac{\delta_u F}{\delta u}\frac{\delta_z u}{\delta z} + \frac{\delta_v F}{\delta v}\frac{\delta_z v}{\delta z} + \frac{\delta_w F}{\delta w}\frac{\delta_z w}{\delta z}$$

So that even in this case we have only three equations as before, by means of which we may determine u, v, w; but since these quantities are themselves functions of x, y, z, we shall not (as in the former case) have determined a point, but a locus; which indeed is as it should be.

Since moreover by means of the operations indicated by the symbol δ n new quantities have been introduced, it will be allowable to assume n relations involving them; let these be that the ratio of the partial variations of any function with respect to any variable to the variation of that variable are equal to the ratios of the corresponding differentials; a definition which will have the double advantage of giving a definite idea of the nature of the variations, and of leaving their absolute magnitudes as arbitrary in this Calculus as those of the differentials were in the Differential Calculus. The analytical expression of the above definition is

$$\frac{\dfrac{d_x\Omega}{dx}}{\dfrac{\delta_x\Omega}{\delta x}} = \frac{\dfrac{d_y\Omega}{dy}}{\dfrac{\delta_y\Omega}{\delta y}} = \frac{\dfrac{d_z\Omega}{dz}}{\dfrac{\delta_z\Omega}{\delta z}} = \ldots = 1, \tag{21}$$

or writing

$$\frac{\delta_x\Omega}{d_x\Omega} = \mathrm{U}, \qquad \frac{\delta_y\Omega}{d_y\Omega} = \mathrm{V}, \qquad \frac{\delta_z\Omega}{d_z\Omega} = \mathrm{W}, \ldots \tag{22}$$

$$\mathrm{U}\frac{dx}{\delta x} = \mathrm{V}\frac{dy}{\delta y} = \mathrm{W}\frac{dz}{\delta z} = \ldots = 1, \tag{23}$$

so that the expression (10) will become

$$\delta\Omega = \frac{d_x\Omega}{dx}\delta x + \frac{d_y\Omega}{dy}\delta y + \frac{d_z\Omega}{dz}\delta z + \ldots \tag{24}$$

a formula which will enable us to calculate the variation of any given function by means of the ordinary processes of the Differential Calculus.

It will be well to show that the variation indicated by the formula (24) does really give rise to a change of form in the function Ω no less than that indicated by (10) did; for in taking the differential (11) of the function, the variables $x, y, z, \ldots$ are by the principles of the Differential Calculus supposed to receive increments consistent with their given connexion; in other words, they are subject to the condition that they must not violate the relation

$$\Omega = 0;$$

and this would still be the case in (24) if the variations $\delta x, \delta y, \delta z, \ldots$ were respectively equal to the differentials $dx, dy, dz, \ldots$; but by (23) the former are shown to be equal to the product of the latter and certain functions of the variables. This must not be supposed to be arguing in a circle; for the expressions

$$\delta_x\Omega, \quad \delta_y\Omega, \quad \delta_z\Omega, \ldots$$

are *by definition* different from

$$d_x\Omega, \quad d_y\Omega, \quad d_z\Omega, \ldots$$

no less than $\delta\Omega$ from $d\Omega$; so that although

$$_1\Omega = f(x + \delta x, \ y + \delta y, \ z + \delta z, \ \ldots) \tag{25}$$

is the same function of

$$x + \delta x, \ y + \delta y, \ z + \delta z, \ \ldots$$

that

$$\Omega_1 = f(x + dx, \ y + dy, \ z + dz, \ \ldots) \tag{26}$$

is of

$$x + dx, \ y + dy, \ z + dz, \ \ldots$$

yet $_1\Omega$ will differ from Ω_1 inasmuch as (4) involve new functions of $x, y, z, \ldots$, which do not appear in (5). It is necessary, however, in order to retain processes in the Calculus of Variations similar to those in that of differentials, to express $_1\Omega$ in the two forms (25) and (26); the latter might have been written also thus,

$$\left.\begin{aligned} {}_1\Omega &= f(x + \mathrm{U}dx, \ y + \mathrm{V}dy, \ z + \mathrm{W}dz, \ \ldots) \\ &= \mathrm{F}(x + dx, \ y + dy, \ z + dz, \ \ldots) \end{aligned}\right\} \tag{27}$$

where F denotes some other function different from f. Hence a change of form does actually take place.

Suppose however that the quantities are subject to both kinds of operations, then by the principles already laid down.

$$d\delta x = \delta(x + dx) - \delta x = \delta dx$$

and so also generally

$$\left.\begin{aligned}\delta^m d^n x &= \delta^{m-1} d\delta d^{n-1} x = \ldots = \delta^{m-1} d^n \delta x \\ &= \delta^{m-2} d\delta d^{n-1}\delta x = \ldots = \delta^{m-2} d^n \delta^2 x \\ &= \ldots\ldots \quad = \ldots = \ldots \\ &= d\delta d^{n-1}\delta^{m-1} x = \ldots = d^n \delta^m x\end{aligned}\right\} \tag{28}$$

and similarly for the other variables $y, z, \ldots$ In order to show that the same relation holds good with respect to any function Ω let

$$\begin{aligned}\Omega_1 &= f(x+dx,\ y+dy,\ z+dz,\ \ldots) \\ \Omega_2 &= f(x+2dx,\ y+2dy,\ z+2dz,\ \ldots) \\ \ldots &= \ldots \\ \Omega_n &= f(x+ndx,\ y+ndy,\ z+ndz,\ \ldots)\end{aligned}$$

and similarly let

$$\begin{aligned}{}_1\Omega &= f(x+\delta x,\ y+\delta y,\ z+\delta z, \ldots) \\ {}_2\Omega &= f(x+2\delta x,\ y+2\delta y,\ z+2\delta z,\ \ldots) \\ \ldots &= \ldots\ldots \\ {}_\nu\Omega &= f(x+\nu\delta x,\ y+\nu\delta y,\ z+\nu\delta z,\ \ldots)\end{aligned}$$

then it is easily seen that

$$\left.\begin{aligned}d^n\Omega &= \Omega_n - \frac{n}{1}\Omega_{n-1} + \frac{n(n-1)}{1.2}\Omega_{n-2} - \ldots = \Sigma(-)^i \frac{n(n-1)\ldots(n-i+1)}{1.2.3\ldots i}\Omega_{n-i} \\ \delta^\nu\Omega &= {}_\nu\Omega - \frac{\nu}{1}{}_{\nu-1}\Omega + \frac{\nu(\nu-1)}{1.2}{}_{\nu-2}\Omega - \ldots = \Sigma(-)^j \frac{\nu(\nu-1)\ldots(\nu-j+1)}{1.2.3\ldots j}{}_{\nu-j}\Omega\end{aligned}\right\} \tag{29}$$

or writing for convenience

$$\begin{gathered}1.2.3\ldots i = \Gamma(i+1), \quad 1.2.3\ldots j = \Gamma(j+1) \\ 1.2.3\ldots n = \Gamma(n+1), \quad 1.2.3\ldots \nu = \Gamma(\nu+1) \\ 1.2.3\ldots(n-i) = \Gamma(n-i+1), \quad 1.2.3\ldots(\nu-j) = \Gamma(\nu-j+1)\end{gathered}$$

so that

$$n(n-1)\ldots(n-i+1) = \frac{\Gamma(n+1)}{\Gamma(n-i+1)}, \quad \nu(\nu-1)\ldots(\nu-j+1) = \frac{\Gamma(\nu+1)}{\Gamma(\nu-j+1)}$$

$$d^n\Omega = \Sigma(-)^i \frac{\Gamma(n+1)}{\Gamma(i+1)\,\Gamma(n-i+1)}\Omega_{n-i}$$

$$\delta^\nu\Omega = \Sigma(-)^j \frac{\Gamma(\nu+1)}{\Gamma(j+1)\,\Gamma(\nu-j+1)}{}_{\nu-j}\Omega$$

so that

$$\delta^\nu d^n\Omega = \Sigma\Sigma(-)^{i+j}\frac{\Gamma(\nu+1)\,\Gamma(n+1)}{\Gamma(j+1)\,\Gamma(\nu-j+1)\,\Gamma(i+1)\,\Gamma(n-i+1)}{}_{\nu-j}[\Omega_{n-i}]$$

$$d^n\delta^\nu\Omega = \Sigma\Sigma(-)^{i+j}\frac{\Gamma(n+1)\,\Gamma(\nu+1)}{\Gamma(i+1)\,\Gamma(n-i+1)\,\Gamma(j+1)\,\Gamma(\nu-j+1)}[{}_{\nu-j}\Omega]_{n-i}$$

but

$$
\begin{aligned}
{}_{\nu-j}[\Omega_{n-i}] &= f\{x+(n-i)dx+(\nu-j)\delta(x+(n-i)dx) \\
&\quad + y+(n-i)dy+(\nu-j)\delta(y+(n-i)dy) \\
&\quad + z+(n-i)dy+(\nu-j)\delta(z+(n-i)dz) + \ldots\} \\
&= f\{x+(\nu-j)\delta x+(n-i)\delta(x+(\nu-j)dx) \\
&\quad + y+(\nu-j)\delta y+(n-i)\delta(y+(\nu-j)dy) \\
&\quad + z+(\nu-j)\delta z+(n-i)\delta(z+(\nu-j)dz) + \ldots\} \\
&= [{}_{\nu-j}\Omega]_{n-i}
\end{aligned}
$$

hence also

$$\delta^{\nu} d^{n}\Omega = d^{n}\delta^{\nu}\Omega \tag{30}$$

as a particular case of this, which is in frequent use, we may here notice

$$n = -1, \ \nu = 1$$

and consequently

$$\delta f\Omega = f\delta\,\Omega \tag{31}$$

II.—GEOMETRICAL INTERPRETATIONS.

In the case of three variables some of the above formulæ admit of elegant geometrical interpretations. The three variables x, y, z will then be the three rectangular co-ordinates of a point on the surface represented by the equation

$$\Omega\,(x, y, z) = 0 \tag{1}$$

Then writing for convenience

$$\frac{d_x\Omega}{dx} = \mathrm{A}, \quad \frac{d_y\Omega}{dy} = \mathrm{B}, \quad \frac{d_z\Omega}{dz} = \mathrm{C}$$

the fundamental formulæ give

$$\left.\begin{aligned} \mathrm{A}dx + \mathrm{B}dy + \mathrm{C}dz = 0 \\ \mathrm{A}\delta z + \mathrm{B}\delta y + \mathrm{C}\delta z = 0 \end{aligned}\right\} \tag{2}$$

whence

$$\frac{\mathrm{A}}{dy\,\delta z - dz\,\delta y} = \frac{\mathrm{B}}{dz\,\delta x - dx\,\delta z} = \frac{\mathrm{C}}{dx\,\delta y - dy\,\delta x} \tag{3}$$

Then in the same manner that

$$\frac{dx}{ds}, \ \frac{dy}{ds}, \ \frac{dz}{ds} \tag{4}$$

(where

$$ds^2 = dx^2 + dy^2 + dz^2)$$

represent the direction-cosines of a line joining the points (x, y, z) with the consecutive point $(x+dx,\ y+dy,\ z+dz)$ on the surface, that is, a tangent line; the expressions

$$\left.\frac{\delta x}{\delta s},\ \frac{\delta y}{\delta s},\ \frac{\delta z}{\delta s}\right\} \tag{5}$$

(where

$$\delta s^2 = \delta x^2 + \delta y^2 + \delta z^2)$$

will represent the direction-cosines of a line joining the point $(x,\ y,\ z)$ on the first surface, with a consecutive point $(x+\delta x,\ y+\delta y,\ z+\delta z)$ on the varied one. And the formulæ (3) show that a transition is made from a point on the first surface to one on the second lying in the tangent plane to the first at the point $x,\ y,\ z$; and that the normal to the first surface is the intersection of two planes whose normals are the two lines whose direction-cosines are (4) and (5) respectively. It is of course obvious that the general theory cannot give the direction of the normal to the second surface, as it would then determine the nature of that surface, but merely shows the law according to which one surface is arrived at from the other. Various other assumptions might have been made instead of (21) of Sect. I. if they had been thought convenient, such as

$$dx\, \delta x + dy\, \delta y + dz\, \delta z = 0$$

in which case the line of transition from one surface to another would always have been perpendicular to the tangent line to the first surface at the point of starting. Or again,

$$\frac{dx}{\delta x} = \frac{dy}{\delta y} = \frac{dz}{\delta z}$$

which would have given as a result that the two lines which in the former case were perpendicular should in this one be parallel, which would be a particular case of the form which was actually chosen.

In the case of two variables, that is of plane curves, (3) becomes

$$\mathrm{A}dx + \mathrm{B}dy = 0$$
$$\mathrm{A}\delta x + \mathrm{B}\delta y = 0$$

whence

$$\frac{dx}{\delta x} = \frac{dy}{\delta y} \text{ or } \frac{dy}{dx} = \frac{\delta y}{\delta x}$$

which is equivalent to the third assumption in the case of three variables. This result might of course have been anticipated à priori, as in plane curves the tangent plane reduces itself to a straight line.

III.—ON THE VARIATION OF A DEFINITE MULTIPLE INTEGRAL.

Consider the multiple integral

$$\Omega = \int_{\mathrm{x}}^{\mathrm{x}'} \int_{\mathrm{y}}^{\mathrm{y}'} \int_{\mathrm{z}}^{\mathrm{z}'} \ldots \Upsilon \ldots dz\, dy\, dx \tag{1}$$

where Υ is any function of the variables

$$x, y, z, \ldots \tag{2}$$

and their differentials of any orders

$$d^l x, d^m y, d^n z, \ldots d^{l-r}x, d^{m-s}y, d^{n-t}z, \ldots \tag{3}$$

Now since, after the integrations indicated in (1) have been performed, Ω will be a function of $\mathrm{x}, \mathrm{x}', \mathrm{y}, \mathrm{y}', \mathrm{z}, \mathrm{z}', \ldots$, and since Ω is a function of Υ, which although a function of the variables (2) and (3), and consequently also of $\mathrm{x}, \mathrm{x}', \mathrm{y}, \mathrm{y}', \mathrm{z}, \mathrm{z}', \ldots$, is still independent of the latter set, we may at once consider Ω as a function of

$$\Upsilon,\ \mathrm{x}, \mathrm{y}, \mathrm{z}, \ldots \mathrm{x}', \mathrm{y}', \mathrm{z}', \ldots \tag{4}$$

hence by the principles laid down in the first section

$$\left.\begin{aligned} \delta\Omega = \frac{d_\Upsilon \Omega}{d\Upsilon}\delta\Upsilon + \frac{d_{\mathrm{x}'}\Omega}{d\mathrm{x}'}\delta\mathrm{x}' + \frac{d_{\mathrm{y}'}\Omega}{d\mathrm{y}'}\delta\mathrm{y}' + \frac{d_{\mathrm{z}'}\Omega}{d\mathrm{z}'}\delta\mathrm{z}' + \ldots \\ + \frac{d_{\mathrm{x}}\Omega}{d\mathrm{x}}\delta\mathrm{x} + \frac{d_{\mathrm{y}}\Omega}{d\mathrm{y}}\delta\mathrm{y} + \frac{d_{\mathrm{z}}\Omega}{d\mathrm{z}}\delta\mathrm{z} + \ldots \end{aligned}\right\} \tag{5}$$

In order to be able to exhibit this expression explicitly, we may remark, that after the integration with respect to x, Ω becomes first a function of x' and then a similar function of x but affected with a negative sign, and similarly with respect to the other variables. Suppose then we represent the value of any function of x, $\phi(x)$ for instance, when x receives a particular value x by the notation

$$\overset{x=\mathrm{x}}{|}\ \phi(x)$$

so that the equation

$$\phi(\mathrm{x}) = \overset{x=\mathrm{x}}{|}\ \phi(x) \tag{6}$$

always holds good. This being premised, the equation may be written in any of the following forms:

$$\left.\begin{aligned} \Omega &= \overset{x=\mathrm{x}'}{|}\ d_x^{-1}\int_{\mathrm{y}}^{\mathrm{y}'}\int_{\mathrm{z}}^{\mathrm{z}'} \ldots \Upsilon \ldots dz\,dy - \overset{x=\mathrm{x}}{|}\ d_x^{-1}\int_{\mathrm{y}}^{\mathrm{y}'}\int_{\mathrm{z}}^{\mathrm{z}'} \ldots \Upsilon \ldots dz\,dy \\ &= \overset{y=\mathrm{y}'}{|}\ d_y^{-1}\int_{\mathrm{x}}^{\mathrm{x}'}\int_{\mathrm{z}}^{\mathrm{z}'} \ldots \Upsilon \ldots dz\,dx - \overset{y=\mathrm{y}}{|}\ d_y^{-1}\int_{\mathrm{x}}^{\mathrm{x}'}\int_{\mathrm{z}}^{\mathrm{z}'} \ldots \Upsilon \ldots dz\,dx \\ &= \overset{z=\mathrm{z}'}{|}\ d_z^{-1}\int_{\mathrm{x}}^{\mathrm{x}'}\int_{\mathrm{y}}^{\mathrm{y}'} \ldots \Upsilon \ldots dy\,dx - \overset{z=\mathrm{z}}{|}\ d_z^{-1}\int_{\mathrm{x}}^{\mathrm{x}'}\int_{\mathrm{y}}^{\mathrm{y}'} \ldots \Upsilon \ldots dy\,dx \end{aligned}\right\} \tag{7}$$

$$= \ldots\ldots$$

It may also be remarked that

$$\left.\begin{aligned}\frac{d_\Upsilon \Omega}{d\Upsilon}\delta\Upsilon &= \frac{d_\Upsilon}{d\Upsilon}\left(\int_{\mathrm{x}}^{\mathrm{x}'}\int_{\mathrm{y}}^{\mathrm{y}'}\int_{\mathrm{z}}^{\mathrm{z}'}\ldots\Upsilon\ldots dz\,dy\,dx\right)\delta\Upsilon \\ &= \int_{\mathrm{x}}^{\mathrm{x}'}\int_{\mathrm{y}}^{\mathrm{y}'}\int_{\mathrm{z}}^{\mathrm{z}'}\ldots\frac{d_\Upsilon\Upsilon}{d\Upsilon}\delta\Upsilon\ldots dz\,dy\,dx \\ &= \int_{\mathrm{x}}^{\mathrm{x}'}\int_{\mathrm{y}}^{\mathrm{y}'}\int_{\mathrm{z}}^{\mathrm{z}'}\ldots\delta\Upsilon\ldots dz\,dy\,dx\end{aligned}\right\} \quad (8)$$

consequently, if the other differentiations indicated in (5) be performed by means of the expressions (7), the final form which the value of $\delta\Omega$ will take will be the following:

$$\begin{aligned}&= \int_{\mathrm{x}}^{\mathrm{x}'}\int_{\mathrm{y}}^{\mathrm{y}'}\int_{\mathrm{z}}^{\mathrm{z}'}\ldots\delta\Upsilon\ldots dz\,dy\,dx \\ &\quad + \Big|^{x=\mathrm{x}'}\int_{\mathrm{y}}^{\mathrm{y}'}\int_{\mathrm{z}}^{\mathrm{z}'}\ldots\Upsilon\ldots dz\,dy\,\delta x - \Big|^{x=\mathrm{x}}\int_{\mathrm{y}}^{\mathrm{y}'}\int_{\mathrm{z}}^{\mathrm{z}'}\ldots\Upsilon\ldots dz\,dy\,\delta x \\ &\quad + \int_{\mathrm{x}}^{\mathrm{x}'}\Big|^{y=\mathrm{y}'}\int_{\mathrm{z}}^{\mathrm{z}'}\ldots\Upsilon\ldots dz\,dx\,\delta y - \int_{\mathrm{x}}^{\mathrm{x}'}\Big|^{y=\mathrm{y}}\int_{\mathrm{z}}^{\mathrm{z}'}\ldots\Upsilon\ldots dz\,dx\,\delta y \\ &\quad + \int_{\mathrm{x}}^{\mathrm{x}'}\int_{\mathrm{y}}^{\mathrm{y}'}\Big|^{z=\mathrm{z}'}\ldots\Upsilon\ldots dy\,dx\,\delta z - \int_{\mathrm{x}}^{\mathrm{x}'}\int_{\mathrm{y}}^{\mathrm{y}'}\Big|^{z=\mathrm{z}}\ldots\Upsilon\ldots dy\,dx\,\delta z \\ &\quad + \ldots\ldots\end{aligned}$$

It is also to be observed that the expression for $\delta\Upsilon$ may be thus written:

$$\begin{aligned}\delta\Upsilon &= \frac{d\Upsilon}{d^{l+1}x}\delta d^l x + \ldots + \frac{d\Upsilon}{d^{l-r+1}x}\delta d^{l-r}x + \ldots + \frac{d\Upsilon}{dx}\delta x \\ &\quad + \frac{d\Upsilon}{d^{m+1}y}\delta d^m y + \ldots + \frac{d\Upsilon}{d^{m-s+1}y}\delta d^{m-s}y + \ldots + \frac{d\Upsilon}{dy}dy \\ &\quad + \frac{d\Upsilon}{d^{n+1}z}\delta d^n z + \ldots + \frac{d\Upsilon}{d^{n-t+1}z}\delta d^{n-t}z + \ldots + \frac{d\Upsilon}{dz}\delta z \\ &\quad + \ldots \\ &= \Sigma_{r=0}^{r=l}\frac{d\Upsilon}{d^{l-r+1}x}d^{l-r}\delta x + \Sigma_{s=0}^{s=m}\frac{d\Upsilon}{d^{m-s+1}y}d^{m-s}\delta y + \Sigma_{t=0}^{t=n}\frac{d\Upsilon}{d^{n-t+1}z}d^{n-t}\delta z + \ldots\end{aligned}$$

where the symbol Σ represents that $r, s, t, \ldots$ receive all values requisite to make the quantities

$$l - r, \quad m - s, \quad n - t, \ldots$$

coincide with the various orders of differentials involved in the function Υ.

Now it is easily seen by successive integration by parts that

$$\left.\begin{aligned}\int \mathrm{L}d^{l-r}\delta x &= S_{\varrho=1}^{\varrho=l-r}(-)^{\varrho-1}d^{\varrho-1}\mathrm{L}d^{l-r-\varrho}\delta x + (-)^{l-r}\int d^{l-r}\mathrm{L}\delta x \\ \int \mathrm{M}d^{m-s}\delta y &= S_{\sigma=1}^{\sigma=m-s}(-)^{\sigma-1}d^{\sigma-1}\mathrm{M}d^{m-s-\sigma}\delta y + (-)^{m-s}\int d^{m-s}\mathrm{M}\delta y \\ \int \mathrm{N}d^{n-t}\delta z &= S_{\tau=1}^{\tau=n-t}(-)^{\tau-1}d^{\tau-1}\mathrm{N}d^{n-t-\tau}\delta z + (-)^{n-t}\int d^{n-t}\mathrm{N}\delta z \\ \ldots\quad & \quad\ldots\end{aligned}\right\} \quad (11)$$

hence, using for convenience the following notation

$$\Big|_{x=\mathrm{x}}^{x=\mathrm{x}'} \phi(x) = \phi(\mathrm{x}') - \phi(\mathrm{x}) \tag{12}$$

the general expression for $\delta\Omega$ may be thus written

$$\left.\begin{array}{l}
\Big|_{x=\mathrm{x}}^{x=\mathrm{x}'} \int_{\mathrm{y}}^{\mathrm{y}'} \int_{\mathrm{z}}^{\mathrm{z}'} \cdot\cdot \left\{ \Sigma_{r=0}^{r=l} \left[S_{\varrho=1}^{\varrho=l-r} (-)^{\varrho-1} d^{\varrho-1} \left(\frac{d\Upsilon}{d^{l-r+1}x} \right) d^{l-r-\varrho}\delta x \right] + \Upsilon \delta x \right\} \cdot\cdot\, dz\, dy \\
+ \int_{\mathrm{x}}^{\mathrm{x}'} \Big|_{y=\mathrm{y}}^{y=\mathrm{y}'} \int_{\mathrm{z}}^{\mathrm{z}'} \cdot\cdot \left\{ \Sigma_{s=0}^{s=m} \left[S_{\sigma=1}^{\sigma=m-s} (-)^{\sigma-1} d^{\sigma-1} \left(\frac{d\Upsilon}{d^{m-s+1}y} \right) d^{m-s-\sigma}\delta y \right] + \Upsilon \delta y \right\} \cdot\cdot\, dz\, dx \\
+ \int_{\mathrm{x}}^{\mathrm{x}'} \int_{\mathrm{y}}^{\mathrm{y}'} \Big|_{z=\mathrm{z}}^{z=\mathrm{z}'} \cdot\cdot \left\{ \Sigma_{t=0}^{t=n} \left[S_{\tau=1}^{\tau=n-t} (-)^{\tau-1} d^{\tau-1} \left(\frac{d\Upsilon}{d^{n-t+1}z} \right) d^{n-t-\tau}\delta z \right] + \Upsilon \delta z \right\} \cdot\cdot\, dy\, dx \\
+ \ldots \\
+ \int_{\mathrm{x}}^{\mathrm{x}'} \int_{\mathrm{y}}^{\mathrm{y}'} \int_{\mathrm{z}}^{\mathrm{z}'} \cdots \left\{ \Sigma_{r=0}^{r=l} \left[(-)^{l-r} d^{l-r} \left(\frac{d\Upsilon}{d^{l-r+1}x} \right) \right] \delta x \right. \\
\qquad + \Sigma_{s=0}^{s=m} \left[(-)^{m-s} d^{m-s} \left(\frac{d\Upsilon}{d^{m-s+1}y} \right) \right] \delta y \\
\qquad \left. + \Sigma_{t=0}^{t=n} \left[(-)^{n-t} d^{n-t} \left(\frac{d\Upsilon}{d^{n-t+1}z} \right) \right] \delta z + \ldots \right\}
\end{array}\right\} \tag{13}$$

IV.—CONDITIONS OF MAXIMA AND MINIMA.

The expression (13) of § II. is of the form

$$\Theta + \int \Phi d\phi = 0 \tag{1}$$

in which the term under the sign of integration by its very nature (since it involves unknown functions) cannot be integrated; but this expression must hold good whatever be the form of the unknown function; i. e. if Φ becomes successively Φ_1, Φ_2, . . ., in other words the equations

$$\Theta + \int \Phi d\phi = 0, \quad \Theta + \int \Phi_1 d\phi = 0, \quad \Theta + \int \Phi_2 d\phi = 0, \ldots$$

must hold good; which obviously involve the conditions

$$\Theta = 0, \qquad \Phi = 0. \tag{2}$$

Hence, in the case of a maximum or minimum we must have

$$\left.\begin{array}{l}
\Big|_{x=\mathrm{x}}^{x=\mathrm{x}'} \int_{\mathrm{y}}^{\mathrm{y}'} \int_{\mathrm{z}}^{\mathrm{z}'} \cdot\cdot \left\{ \Sigma_{r=0}^{r=l} \left[S_{\varrho=1}^{\varrho=l-r} (-)^{\varrho-1} d^{\varrho-1} \left(\frac{d\Upsilon}{d^{l-r+1}x} \right) d^{l-r-\varrho}\delta x \right] + \Upsilon \delta x \right\} \cdot\cdot\, dz\, dy \\
+ \int_{\mathrm{x}}^{\mathrm{x}'} \Big|_{y=\mathrm{y}}^{y=\mathrm{y}'} \int_{\mathrm{z}}^{\mathrm{z}'} \cdot\cdot \left\{ \Sigma_{s=0}^{s=m} \left[S_{\sigma=1}^{\sigma=m-s} (-)^{\sigma-1} d^{\sigma-1} \left(\frac{d\Upsilon}{d^{m-s+1}y} \right) d^{m-s-\sigma}\delta y \right] + \Upsilon \delta y \right\} \cdot\cdot\, dz\, dx \\
+ \int_{\mathrm{x}}^{\mathrm{x}'} \int_{\mathrm{y}}^{\mathrm{y}'} \Big|_{z=\mathrm{z}}^{z=\mathrm{z}'} \cdot\cdot \left\{ \Sigma_{t=0}^{t=n} \left[S_{\tau=1}^{\tau=n-t} (-)^{\tau-1} d^{\tau-1} \left(\frac{d\Upsilon}{d^{n-t+1}z} \right) d^{n-t-\tau}\delta z \right] + \Upsilon \delta z \right\} \cdot\cdot\, dy\, dx \\
\ldots = 0
\end{array}\right\} \tag{3}$$

and

$$\left.\begin{aligned}&\Sigma_{r=0}^{r=l}\Big[(-)^{l-r}d^{l-r}\Big(\frac{d\Upsilon}{d^{l-r+1}x}\Big)\Big]\delta x\\ &+\Sigma_{s=0}^{s=m}\Big[(-)^{m-s}d^{m-s}\Big(\frac{d\Upsilon}{d^{m-s+1}y}\Big)\Big]\delta y\\ &+\Sigma_{t=0}^{t=n}\Big[(-)^{n-t}d^{n-t}\Big(\frac{d\Upsilon}{d^{n-t+1}z}\Big)\Big]\delta z\\ &+\ \ldots = 0\end{aligned}\right\}\qquad(4)$$

the latter of which equations separates itself into the following system

$$\left.\begin{aligned}&\Sigma_{r=0}^{r=l}\Big[(-)^{-r}d^{l-r}\Big(\frac{d\Upsilon}{d^{l-r+1}x}\Big)\Big]=0,\qquad \Sigma_{s=0}^{s=m}\Big[(-)^{-s}d^{m-s}\Big(\frac{d\Upsilon}{d^{m-s+1}z}\Big)\Big]=0,\\ &\qquad\Sigma_{t=0}^{t=n}\Big[(-)^{-t}d^{n-t}\Big(\frac{d\Upsilon}{d^{n-t+1}z}\Big)\Big]=0,\ldots.\end{aligned}\right\}\qquad(5)$$

Suppose that besides the function Ω the variables are subject to certain conditions, such that Ω is to be a maximum or minimum for given values of certain of the functions of the variables, as

$$\mathrm{U},\ \mathrm{V},\ \mathrm{W},\ \ldots \qquad (6)$$

and let the given values be represented by the equations

$$\mathrm{U} = a,\ \mathrm{V} = b,\ \mathrm{W} = c,\ \ldots \qquad (7)$$

then whatever supposition makes Ω a maximum or minimum, i. e.

$$\delta\Omega = 0 \qquad (8)$$

must also make

$$\delta\mathrm{U} = 0,\ \delta\mathrm{V} = 0,\ \delta\mathrm{W} = 0,\ \ldots \qquad (9)$$

But equations (3) and (4) are equivalent to the following

$$\left.\begin{aligned}&\frac{d\Omega}{dx}\delta x + \frac{d\Omega}{dy}\delta y + \frac{d\Omega}{dz}\delta z + \ldots = 0\\ &\frac{d\mathrm{U}}{dx}\delta x + \frac{d\mathrm{U}}{dy}\delta y + \frac{d\mathrm{U}}{dz}\delta z + \ldots = 0\\ &\frac{d\mathrm{V}}{dx}\delta x + \frac{d\mathrm{V}}{dy}\delta y + \frac{d\mathrm{V}}{dz}\delta z + \ldots = 0\\ &\frac{d\mathrm{W}}{dx}\delta x + \frac{d\mathrm{W}}{dy}\delta y + \frac{d\mathrm{W}}{dz}\delta z + \ldots = 0\\ &\ldots\end{aligned}\right\}\qquad(10)$$

Now by methods exactly similar to those employed in the differential calculus will give us by means of Lagrange's theory of indeterminate multipliers the following equations (in which λ, μ, ν, . . . are the multipliers),

$$\left.\begin{aligned}
&\frac{d\Omega}{dx} + \lambda\frac{dU}{dx} + \mu\frac{dV}{dx} + \nu\frac{dW}{dx} + \ldots = 0 \\
&\frac{d\Omega}{dy} + \lambda\frac{dU}{dy} + \mu\frac{dV}{dy} + \nu\frac{dW}{dy} + \ldots = 0 \\
&\frac{d\Omega}{dz} + \lambda\frac{dU}{dz} + \mu\frac{dV}{dz} + \nu\frac{dW}{dz} + \ldots = 0 \\
&\ldots
\end{aligned}\right\} \qquad (11)$$

or multiplying these by δx, δy, δz, . . . respectively, and adding, the general condition becomes

$$\delta\Omega + \lambda\delta U + \mu\delta V + \nu\delta W + \ldots = 0 \qquad (12)$$

that is

$$\delta(\Omega + \lambda U + \mu V + \nu W + \ldots) = 0$$

the usual formula for the investigation of Isoperimetrical Problems.

It remains for us to determine the conditions which will show whether the result obtained by the above rules be a maximum or minimum; for this purpose it will be necessary to have recourse to the second variation of the function Ω.

The conditions relative to the case in which the limits are subject to variation presents some difficulty, we shall therefore for the present content ourselves with determining the criteria when the variations of the limits are not taken into account; in this case we have the principles already laid down,

$$\left.\begin{aligned}
\delta\Omega = \int_{x}^{x'}\int_{y}^{y'}\int_{z}^{z'}\cdots\Big[&\Sigma_{r=0}^{r=l}\frac{d\Upsilon}{d^{l-r+1}x}\,d^{l-r}\delta x + \Sigma_{s=0}^{s=m}\frac{d\Upsilon}{d^{m-s+1}y}\,d^{m-s}\delta y \\
&+ \Sigma_{t=0}^{t=n}\frac{d\Upsilon}{d^{n-t+1}z}\,d^{n-t}\delta z + \ldots\Big]
\end{aligned}\right\} \qquad (13)$$

so that the second variation may be written

$$\left.\begin{aligned}
\delta^2\Omega = \int_{x}^{x'}\int_{y}^{y'}\int_{z}^{z'}\cdots\Big[&\Sigma_{r=0}^{r=l}\frac{d^2\Upsilon}{d^{l-r+2}x}(d^{l-r}\delta x)^2 \\
&+ \Sigma_{s=0}^{s=m}\frac{d^2\Upsilon}{d^{m-s+2}y}(d^{m-s}\delta y)^2 \\
&+ \Sigma_{t=0}^{t=n}\frac{d^2\Upsilon}{d^{n-t+2}z}(d^{n-t}\delta z)^2 \\
&+ 2\Sigma_{s=0}^{s=m}\Sigma_{t=0}^{t=n}\frac{d^2\Upsilon}{d^{m-s+1}y\,d^{n-t+1}z}\,d^{m-s+1}\delta y\,d^{n-t+1}\delta z \\
&+ 2\Sigma_{t=0}^{t=n}\Sigma_{r=0}^{r=l}\frac{d^2\Upsilon}{d^{n-t+1}z\,d^{l-r+1}x}\,d^{n-t+1}\delta z\,d^{l-r+1}\delta x \\
&+ 2\Sigma_{r=0}^{r=l}\Sigma_{s=0}^{s=m}\frac{d^2\Upsilon}{d^{l-r+1}x\,d^{m-s+1}y}\,d^{l-r+1}\delta x\,d^{m-s+1}\delta y \\
&+ \ldots\Big]
\end{aligned}\right\} \qquad (14)$$

that is of the form

$$\int(\mathrm{A}u^2+\mathrm{B}v^2+\mathrm{C}w^2+\ \ldots\ +2\mathrm{F}vw+2\mathrm{G}wu+2\mathrm{H}uv+\ \ldots) \tag{15}$$

Now if the element under the sign of integration remains always of the same sign, the criteria will be found in the same way as those relating to ordinary maxima and minima of many variables. But if this be not the case, the element in question may be divided into certain parts, one of which remains always of the same sign, and the remainder are integrable; the integrals of the latter will contain the variations $\delta x, \delta y, \delta z, \ldots$ or their differentials as factors, which will vanish at the limits; the integrals themselves will consequently also vanish, and the whole will be reduced to an integral, the element of which, under the sign of integration, remains always of the same sign. In the case under consideration the part integrated, with respect to x, will be generally of the form

$$au^2+bv^2+cw^2+\ \ldots\ +2fvw+2gwu+2huv+\ \ldots$$

that with respect to y,

$$a'u^2+b'v^2+c'w^2+\ \ldots +2f'vw+2g'wu+2h'uv+\ \ldots$$

that with respect to z,

$$a''u^2+b''v^2+c''w^2+\ \ldots\ +2f''vw+2g'wu+2h'uv+\ \ldots$$

and so on. So that differentiating the first of these with respect to x, the second with respect to y, the third with respect to z, and so on, and subtracting the results from the element of the second variation of Ω, that is from (15), and writing for convenience

$$\left.\begin{array}{lll} \mathfrak{A}=\mathrm{A}-\dfrac{da}{dx}-\dfrac{da'}{dy}-\dfrac{da''}{dz}\ldots, & \mathfrak{B}=\mathrm{B}-\dfrac{db}{dx}-\dfrac{db'}{dy}-\dfrac{db''}{dz}\ldots, & \mathfrak{C}=\mathrm{C}-\dfrac{dc}{dx}-\dfrac{dc'}{dy}-\dfrac{dc''}{dz}\ldots \\ \mathfrak{F}=\mathrm{F}-\dfrac{df}{dx}-\dfrac{df'}{dy}-\dfrac{df''}{dz}\ldots, & \mathfrak{G}=\mathrm{G}-\dfrac{dg}{dx}-\dfrac{dg'}{dy}-\dfrac{dg''}{dz}\ldots, & \mathfrak{H}=\mathrm{H}-\dfrac{dh}{dx}-\dfrac{dh'}{dy}-\dfrac{dh''}{dz}\ldots \\ \ldots & \ldots & \ldots \end{array}\right\} \tag{16}$$

the element becomes

$$\mathfrak{A}u^2+\mathfrak{B}v^2+\mathfrak{C}w^2+\ \ldots\ +2\mathfrak{F}vw+2\mathfrak{G}wu+2\mathfrak{H}uv+\ \ldots\ =\Theta \tag{17}$$

which must remain always of the same sign. To determine the conditions that this may be the case we will follow the course taken by Lagrange in the *Théorie des Fonctions Analytiques.* Suppose it must remain aways positive, it is evident that, for the contrary case, it will be sufficient to take the quantities $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}, \ldots$, affected with a negative sign. Since then it must never become negative, it follows that it should have a positive minimum; and conversely, if it has only positive minima it can never become negative. It remains only to find the conditions that the quantity under consideration shall have all its minima positive. Following the method of ordinary maxima and minima, we shall find the first and second differential coefficients of the quantity Θ with respect to one variable, as u, and equate the first to zero, and suppose the second positive. That is

$$\left.\begin{array}{c}\mathfrak{A}u+\mathfrak{H}v+\mathfrak{G}w+\ldots=0\\ \mathfrak{A}>0\end{array}\right\}\qquad(18)$$

Substituting the value of u, deduced from the former of these equations in the value of Θ, and writing for convenience

$$\left.\begin{array}{lll}\mathfrak{L}=\mathfrak{B}-\dfrac{\mathfrak{H}^2}{\mathfrak{A}}, & \mathfrak{M}=\mathfrak{C}-\dfrac{\mathfrak{G}^2}{\mathfrak{A}}, & \mathfrak{N}=\\ \mathfrak{P}=\mathfrak{F}-\dfrac{\mathfrak{G}\mathfrak{H}}{\mathfrak{A}}, & \ldots & \ldots\\ \ldots & \ldots & \ldots\end{array}\right\}\qquad(19)$$

the expression for Θ becomes

$$\Theta=\mathfrak{L}v^2+\mathfrak{M}w^2+\mathfrak{N}r^2+\ldots+2\mathfrak{P}wr+2\mathfrak{Q}rv+2\mathfrak{R}vw+\ldots\qquad(20)$$

In the same manner as before, finding the first and second differential coefficients of this expression with respect to one variable, as v, the conditions become

$$\left.\begin{array}{c}\mathfrak{L}v^2+\mathfrak{R}w+\mathfrak{Q}r+\ldots=0\\ \mathfrak{L}>0\end{array}\right\}\qquad(21)$$

Similarly, the expression for Θ may be again transformed by means of these equations into another of the same form, such as

$$\Theta=\mathfrak{U}w^2+\mathfrak{V}r^2+\mathfrak{W}s^2+\ldots 2\mathfrak{X}rs+2\mathfrak{Y}sw+2\mathfrak{Z}wr+\ldots\qquad(22)$$

and the conditions again become

$$\left.\begin{array}{c}\mathfrak{U}w+\mathfrak{Z}r+\mathfrak{Y}s+\ldots=0\\ \mathfrak{U}>0\end{array}\right\}\qquad(23)$$

and so on. Now it is easily seen that the last of these transformed expressions, which will contain only one of the quantities $u, v, w, \ldots$ and which will be of the form

$$\Phi t^2$$

will be the minimum of the quantity Θ; hence the conditions that Θ shall have a positive minimum will be

$$\mathfrak{A}>0,\quad \mathfrak{L}>0,\quad \mathfrak{W}>0,\ldots\quad \Phi>0\qquad(24)$$

and since the equations which determine $u, v, w, \ldots$ are of the first degree, we may conclude that that will be the only minimum; hence the problem is completely solved.

It may also be remarked that by the above transformations the value of Θ becomes

$$\left.\begin{aligned} &\mathfrak{A}\left(u + \frac{\mathfrak{H}v + \mathfrak{G}w + \ldots}{\mathfrak{A}}\right)^2 \\ &+ \mathfrak{L}\left(v + \frac{\mathfrak{R}w + \mathfrak{Q}r + \ldots}{\mathfrak{L}}\right)^2 \\ &+ \mathfrak{U}\left(w + \frac{\mathfrak{Z}r + \mathfrak{Y}s + \ldots}{\mathfrak{U}}\right)^2 + \ldots \end{aligned}\right\} \qquad (25)$$

from which expression also it appears that the conditions (24) will ensure that the element (17) shall always remain positive.

Problems in the Calculus of Variations.

The following problems are collected from various sources.

(I.) To find the shortest line in space from one given curve to another.

We must find

$$\int ds = \text{a minimum.} \tag{1}$$

$$\therefore \quad \delta \int ds = 0 \tag{2}$$

but

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{3}$$

$$\therefore \quad \delta ds = \frac{dx d\delta x + dy d\delta y + dz d\delta z}{ds} \tag{4}$$

$$\left.\begin{aligned} \therefore \quad \int \delta ds = & \left[\frac{dx}{ds}\delta x + \frac{dy}{ds}\delta y + \frac{dz}{ds}\delta z\right]_{x'y'z'}^{x''y''z''} \\ & - \int \left(\delta x d\frac{dx}{ds} + \delta y d\frac{dy}{ds} + \delta z d\frac{dz}{ds}\right) = 0 \end{aligned}\right\} \tag{5}$$

where $x'\,y'\,z'$ are the co-ordinates of one curve, and $x''\,y''\,z''$ those of the other. And since the curves are entirely independent of one another, the first part gives

$$\frac{dx''}{ds''}\delta x'' + \frac{dy''}{ds''}\delta y'' + \frac{dz''}{ds''}\delta z'' = 0 \tag{6}$$

$$\frac{dx'}{ds'}dx' + \frac{dy'}{ds'}dy' + \frac{dz'}{ds'}\delta z' = 0 \tag{7}$$

Now, $\frac{dx''}{ds''}\,\frac{dy''}{ds''}\,\frac{dz''}{ds''}$ are the direction-cosines of a tangent to one curve, and $\frac{dx'}{ds'}\,\frac{dy'}{ds'}\,\frac{dz'}{ds'}$ those of a tangent to the other, therefore (6) and (7) show that the required curve cuts both the given curves at right angles.

The second part, since δx, δy, δz are now absolutely arbitrary, gives

$$d\frac{dx}{ds} = 0 \qquad d\frac{dy}{ds} = 0 \qquad d\frac{dz}{ds} = 0 \tag{8}$$

$$\therefore \quad \frac{dx}{ds} = l \qquad \frac{dy}{ds} = m \qquad \frac{dz}{ds} = n \tag{9}$$

which shows that the line must be straight, its equations being

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}$$

(II.) If it were required to draw the shortest line from one given point to another, the first part would equal zero of itself, and the second would show that the line must be straight.

(III.) To find the shortest line on a given surface between two given points.

Let
$$u = 0 \tag{1}$$
be the equation to the surface.

Consequently, as before, we must have
$$\delta \int ds = 0 \tag{2}$$
and also
$$\left.\begin{aligned} &\frac{dx}{ds}\delta x + \frac{dy}{ds}\delta y + \frac{dz}{ds}\delta z \\ &-\int \left\{ \delta x d\frac{dx}{ds} + \delta y d\frac{dy}{ds} + \delta z d\frac{dz}{ds} \right\} = 0 \end{aligned}\right\} \tag{3}$$

Now since δx, δy, δz must vanish at the fixed limits, therefore the first part will equal zero of itself; and in order that the integral may vanish we must have
$$d\left(\frac{dx}{ds}\right)\delta x + d\left(\frac{dy}{ds}\right)\delta y + d\left(\frac{dz}{ds}\right)\delta z = 0 \tag{4}$$
whatever be the values of δx, δy, δz, so that they satisfy the condition that the point (x, y, z) is on the given surface; viz.
$$\frac{du}{dx}\delta x + \frac{du}{dy}\delta y + \frac{du}{dz}\delta z = 0 \tag{5}$$
consequently we must have
$$\frac{d\,.\,\frac{dx}{ds}}{\frac{du}{dx}} = \frac{d\,.\,\frac{dy}{ds}}{\frac{du}{dy}} = \frac{d\,.\,\frac{dz}{ds}}{\frac{du}{dz}} \tag{6}$$

We will now show that the osculating plane to the curve is always a normal plane to the surface,

Let
$$\left.\begin{aligned} \mathrm{P} &= dyd^2z - dzd^2y \\ \mathrm{Q} &= dzd^2x - dxd^2z \\ \mathrm{R} &= dxd^2y - dyd^2x \end{aligned}\right\} \tag{7}$$
then, the perpendicular on the osculating plane makes angles with the co-ordinate axes, whose cosines are P, Q, R divided respectively by
$$\sqrt{\mathrm{P}^2 + \mathrm{Q}^2 + \mathrm{R}^2}$$

Now,

$$d\frac{dx}{ds} = \frac{dsd^2x - dxd^2s}{ds^2} = \frac{ds^2d^2x - dxdsd^2s}{ds^3}$$

$$= \frac{1}{ds^3}\left\{(dx^2 + dy^2 + dz^2)d^2x - dx(dxd^2x + dyd^2y + dzd^2z)\right\}$$

$$= \frac{1}{ds^3}\left\{dz(dzd^2x - dxd^2z) - dy(dxd^2y - dyd^2x)\right\}$$

$$= \frac{1}{ds^3}(\mathrm{Q}dz - \mathrm{R}dy)$$

Similarly

$$\left.\begin{aligned} d\frac{dy}{ds} &= \frac{1}{ds^3}\left\{\mathrm{R}dx - \mathrm{P}dz\right\} \\ d\frac{dz}{ds} &= \frac{1}{ds^3}\left\{\mathrm{P}dy - \mathrm{Q}dx\right\} \end{aligned}\right\} \tag{8}$$

and therefore identically

$$\mathrm{P}d\frac{dx}{ds} + \mathrm{Q}d\frac{dy}{ds} + \mathrm{R}d\frac{dz}{ds} = 0 \tag{9}$$

and also by (6)

$$\mathrm{P}\frac{du}{dx} + \mathrm{Q}\frac{du}{dy} + \mathrm{R}\frac{du}{dz} = 0 \tag{10}$$

i. e. the normal to the surface is perpendicular to a perpendicular to the osculating plane of the curve, i. e. coincides with the osculating plane. Q. E. D.

(IV.) Find the curve which of all that can be drawn between two given points contains between the evolute and the radii of curvature at its extremities the greatest area.

$$\text{Area} = \tfrac{1}{2}\int \rho ds$$

where

$$\rho = \frac{ds^3}{dxd^2y - dyd^2x}$$

hence

$$\int(\rho d\delta s + ds\delta\rho) = 0$$

and

$$d\delta s = \frac{dx}{ds}d\delta x + \frac{dy}{ds}d\delta y$$

$$\delta\rho = -\frac{\rho^2}{ds^3}(dxd^2\delta y - d^2xd\delta y - dyd^2\delta x + d^2yd\delta x) + \frac{3\rho}{ds}d\delta s$$

hence

$$\int\left[4\rho d\delta s - \frac{\rho^2}{ds^2}(dxd^2\delta y - d^2xd\delta y - dyd^2\delta x + d^2yd\delta x)\right] = 0$$

or

$$\int\Big[\rho\Big(4\frac{dx}{ds}-\frac{\rho}{ds^2}d^2y\Big)d\delta x+\frac{\rho}{ds}\Big(4dy+\frac{\rho}{ds}d^2x\Big)d\delta y$$
$$+\frac{\rho^2}{ds^2}dyd^2\delta x-\frac{\rho^2}{ds^2}dxd^2\delta y\Big]=0$$

hence, integrating by parts,

$$\Big[\frac{\rho}{ds}\Big(4dx-\frac{\rho}{ds}d^2y\Big)-d\Big(\frac{\rho^2}{ds^2}dy\Big)\Big]\delta x+\Big[\frac{\rho}{ds}\Big(4dy+\frac{\rho}{ds}d^2x\Big)+d\Big(\frac{\rho^2}{ds^2}dz\Big)\Big]\delta y$$
$$+\frac{\rho^2}{ds^2}dgd\delta x-\frac{\rho^2}{ds^2}dxd\delta y$$
$$-\int\Big[\Big\{d\Big(\frac{\rho}{ds}(4dx-\frac{\rho}{ds}d^2y)\Big)-d^2\Big(\frac{\rho^2}{ds^2}dy\Big)\Big\}\delta x$$
$$+\Big\{d\Big(\frac{\rho}{ds}(4dy+\frac{\rho}{ds}d^2x)\Big)+d^2\Big(\frac{\rho^2}{ds^2}dx\Big)\Big\}\delta y\Big]=0$$

hence

$$\frac{\rho}{ds}\Big(4dx-\frac{\rho}{ds}d^2y\Big)-d\Big(\frac{\rho^2}{ds^2}dy\Big)=2a$$
$$\frac{\rho}{ds}\Big(4dy+\frac{\rho}{ds}d^2x\Big)+d\Big(\frac{\rho^2}{ds^2}dx\Big)=2b$$

or

$$\frac{\rho}{ds}\Big(4dx-\frac{2\rho}{ds}d^2y\Big)-dyd\Big(\frac{\rho^2}{ds^2}\Big)=2a$$
$$\frac{\rho}{ds}\Big(4dy+\frac{2\rho}{ds}d^2x\Big)+dxd\Big(\frac{\rho^2}{ds^2}\Big)=2b$$

hence

$$2\rho ds+\frac{\rho^2}{ds^2}(d^2xdy-d^2ydx)=adx+bdy=\rho ds$$

that is, taking x as the independent variable,

$$1=\frac{a\dfrac{d^2y}{dx^2}}{\Big(1+\dfrac{dy^2}{dx^2}\Big)^2}+\frac{b\dfrac{dy}{dx}\dfrac{d^2y}{dx^2}}{\Big(1+\dfrac{dy^2}{dx^2}\Big)^2}$$

or integrating

$$x-\mathrm{x}=\frac{1}{2}\frac{ax+b}{1+\dfrac{dy^2}{dx^2}}+\frac{1}{2}a\tan^{-1}\frac{dy}{dx}$$

which is not integrable; but if $a=0$

$$\frac{1}{2}\frac{b}{x-\mathrm{x}}=1+\frac{dy^2}{dx^2}\qquad\frac{dy}{dx}=\sqrt{\frac{1}{2}\frac{b}{x-\mathrm{x}}-1}$$

the equation to a cycloid.

(V.) To find the relation between x and y, so that

$$\int(x^2+y^2)^{\frac{n}{2}}ds \text{ may be a minimum,}$$

changing to polar co-ordinate

$$\delta\int r^n\sqrt{dr^2+r^2d\theta^2}=0$$

$$\therefore\quad \int nr^{n-1}\delta r ds+r^n\delta ds=0$$

$$ds^2=dr^2+r^2d\theta^2\quad \therefore\ \delta ds=\frac{dr}{ds}\delta dr+\frac{rd\theta^2}{ds}\delta r+r^2\frac{d\theta}{ds}\delta d\theta$$

whence, integrating by parts, we have

$$r^n\left\{\frac{dr}{ds}\delta r+r^2\frac{d\theta}{ds}\delta\theta\right\}+\int\left\{\left(nr^{n-1}ds+r^{n+1}\frac{d\theta^2}{ds}-dr\left(r^n\frac{dr}{ds}\right)\right)\delta r-d\left(r^{n+2}\frac{d\theta}{ds}\right)\delta\theta\right\}=0$$

between the assigned limits,

$$\therefore\quad nr^{n-1}ds+r^{n+1}\frac{d\theta^2}{ds}-d\left(r^n\frac{dr}{ds}\right)=0 \tag{1}$$

and

$$r^{n+2}\frac{d\theta}{ds}=0 \tag{2}$$

from either of which we obtain the equation

$$\frac{kdr}{r\sqrt{r^{2n+2}-k^2}}=\pm\,d\theta$$

which being rationalized and integrated give

$$r^{n+1}=k\sec\{\overline{n+1}(\theta+\mathrm{C})\}$$

whence we know the relation between x and y.

If $n=0$ we have the equation to a straight line.

(VI.) To find the curve of quickest descent from one given curve to another.

$$t=\int\frac{ds}{\sqrt{2g}\sqrt{h-z}}$$

$$\therefore\ \delta\int\frac{ds}{\sqrt{h-z}}=0$$

$$\therefore\int\left\{\frac{1}{\sqrt{h-z}}\delta ds+\frac{1}{2}\frac{ds}{(h-z)^{\frac{3}{2}}}\delta z\right\}=0$$

$$\therefore\int\left\{\frac{1}{\sqrt{h-z}}\left(\frac{dx}{ds}d\delta x+\frac{dy}{ds}d\delta y+\frac{dz}{ds}d\delta z\right)+\frac{1}{2}\frac{ds}{(h-z)^{\frac{3}{2}}}\delta z\right\}=0$$

therefore integrating by parts,

$$\frac{1}{\sqrt{h-z}}\left(\frac{dx}{ds}\delta x+\frac{dy}{ds}\delta y+\frac{dz}{ds}\delta z\right)$$
$$-\int\left\{d\left(\frac{dx}{ds}\frac{1}{\sqrt{h-z}}\right)\delta x+d\left(\frac{dy}{ds}\frac{1}{\sqrt{h-z}}\right)\delta y\right.$$
$$\left.+\left[d\left(\frac{dz}{ds}\frac{1}{\sqrt{h-z}}\right)-\frac{1}{2}\frac{ds}{(h-z)^{\frac{3}{2}}}\right]\delta z\right\}=0$$

The first part shows, as before, that the curve of quickest descent must cut the given curves at right angles ; from the second part we get

$$d\left(\frac{dx}{ds}\frac{1}{\sqrt{h-z}}\right) = 0 \qquad d\left(\frac{dy}{ds}\frac{1}{\sqrt{h-z}}\right) = 0$$

$$\therefore \quad \frac{dx}{ds}\frac{1}{\sqrt{h-z}} = a \qquad \frac{dy}{ds}\frac{1}{\sqrt{h-z}} = b$$

$$\therefore \quad \frac{dy}{dx} = \frac{b}{a} \text{ a constant quantity;}$$

consequently the required curve lies wholly in one plane. Let this be the plane of $x\,z$.

$$\therefore \quad \frac{dx}{ds} = a\,.\sqrt{h-z}$$

$$\frac{ds^2}{dx^2} = \frac{1}{a^2(h-z)} = 1 + \frac{dz^2}{dx^2}$$

$$\therefore \quad \frac{dz^2}{ds^2} = \frac{1-a^2h+a^2z}{a^2h-a^2z} \text{ a cycloid.}$$

If the motion commences from the origin, instead of

$$h-z$$

we must write

$$z$$

We have not written down the limits, as the method is the same as in the other problems ; and the problem

(VII.) To find the curve of quickest descent from one given point to another, being only a particular case of this, must be treated in the same way.

(VIII.) A particle is in motion under the action of a central force, find the nature of the orbit described by the principle of least action.

$$\int v ds = \text{minimum}$$

Now

$$v^2 = C - 2\int F dr$$

$$\therefore \quad v = \sqrt{C-2\int F dr}$$

$$ds = \sqrt{r^2 + \frac{dr^2}{d\theta^2}}\,d\theta$$

$$\therefore \quad u = \int\sqrt{\left(C-2\int F dr\right)\left(r^2+\frac{dr^2}{d\theta^2}\right)}\,d\theta$$

and taking the variation with respect to θ only, as the coefficients of each of the variations δr $\delta\theta$ must be put $=0$, we get

$$\delta u = 0 = \int\left\{\sqrt{\left(C-2\int F dr\right)\left(r^2+\frac{dr^2}{d\theta^2}\right)}\, d\delta\theta \right.$$

$$\left. + \sqrt{C-2\int F dr}\,.\,d\theta \frac{-\dfrac{dr^2}{d\theta}\cdot\dfrac{d\delta\theta}{d\theta^2}}{\sqrt{r^2+\dfrac{dr^2}{d\theta^2}}} + \dots\right\}$$

therefore integrating by parts

$$\left(\sqrt{\left(C-2\int F dr\right)\left(r^2+\frac{dr^2}{d\theta^2}d\theta\right)} - \frac{\dfrac{dr^2}{d\theta^2}}{\sqrt{r^2+\dfrac{dr^2}{d\theta^2}}}\right)\delta\theta$$

$$-\int\left\{\left(d\sqrt{\left(C-2\int F dr\right)\left(r^2+\frac{dr^2}{d\theta^2}\right)} - d.\frac{\sqrt{C-2\int F dr}}{\sqrt{r^2+\dfrac{dr^2}{d\theta^2}}}\frac{dr^2}{d\theta^2}\right)\delta\theta + \dots\right\} = 0$$

$$\therefore\ d\left\{\sqrt{\left(C-2\int F dr\right)\left(r^2+\frac{dr^2}{d\theta^2}\right)} - \frac{\sqrt{C-2\int F dr}}{\sqrt{r^2+\dfrac{dr^2}{d\theta^2}}}\frac{dr^2}{d\theta^2}\right\} = 0$$

$$\sqrt{C-2\int F dr}\left(r^2+\frac{dr^2}{d\theta^2}\right) - \sqrt{C-2\int F dr}\,.\,\frac{dr^2}{d\theta^2} = h\sqrt{r^2+\frac{dr^2}{d\theta^2}}$$

$$\therefore\ r^2\sqrt{C-2\int F dr} = h\sqrt{r^2+\frac{dr^2}{d\theta^2}}$$

$$\therefore\ C-2\int F dr = \frac{h^2}{r^2} + \frac{h^2}{r^2}\frac{dr^2}{d\theta^2}$$

Or let

$$r = \frac{1}{u}$$

$$\therefore\ C+2\int F\frac{du}{u^2} = h^2\left(u^2+\frac{du^2}{d\theta^2}\right)$$

which may be applied to particular cases, and is the usual formula.

(IX.) A string is stretched over a given surface; show that it will place itself along the shortest path, if the resistance of the surface be the only acting force.

Let

$$u = 0$$

be the equation to the surface.

Let X, Y, Z be the resolved parts of the forces acting at the point P (x, y, z).

Let

$$T = \text{tension at P}$$
$$T' = T + \Delta T \text{ . . . . . at P}'$$

consequently the sum of the forces acting on PP′ parallel to the axis of x will be

$$\int_{s}^{s'} X ds + T' \frac{dx'}{ds'} - T \frac{dx}{ds} = 0$$

and also when the element is indefinitely small the three components will be

$$X ds + d\left(T \frac{dx}{ds}\right) = 0$$
$$Y ds + d\left(T \frac{dy}{ds}\right) = 0$$
$$Z ds + d\left(T \frac{dz}{ds}\right) = 0$$

and in the present case

$$R = \text{pressure or resistance of surface.}$$

Let

$$V = \left\{ \left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2 \right\}^{\frac{1}{2}}$$

$$\left.\begin{aligned} \therefore\ X &= R \frac{\frac{du}{dx}}{V} \\ Y &= R \frac{\frac{du}{dy}}{V} \\ Z &= R \frac{\frac{du}{dx}}{V} \end{aligned}\right\}$$

$$\therefore\ \left.\begin{aligned} R \frac{\frac{du}{dx}}{V} ds + d\left(T . \frac{dx}{ds}\right) &= 0 \\ R \frac{\frac{du}{dy}}{V} ds + d\left(T . \frac{dy}{ds}\right) &= 0 \\ R . \frac{\frac{du}{dz}}{V} ds + d\left(T \frac{dz}{ds}\right) &= 0 \end{aligned}\right\}$$

Multiplying by $\frac{dx}{ds}\frac{dy}{ds}\frac{dz}{ds}$ respectively, and adding, we get

$$dT + T\left(\frac{dx}{ds}d.\frac{dx}{ds} + \frac{dy}{ds}d.\frac{dy}{ds} + \frac{dz}{ds}d.\frac{dz}{ds}\right) = 0$$

but

$$\frac{dx^2}{ds^2} + \frac{dy^2}{ds^2} + \frac{dz^2}{ds^2} = 1$$

$$\therefore \quad \frac{dx}{ds}d\frac{dx}{ds} + \frac{dy}{ds}d\frac{dy}{ds} + \frac{dz}{ds}d\frac{dz}{ds} = 0$$

$$\therefore \quad dT = 0$$

$$\therefore \quad T = \text{constant.}$$

$$\therefore \quad \left.\begin{aligned} \frac{R\frac{du}{dx}}{V}ds &= -T\,d\frac{dx}{ds} \\ \frac{R\frac{du}{dy}}{V}ds &= -T\,d\frac{dy}{ds} \\ \frac{R\frac{du}{dz}}{V}ds &= -T\,d\frac{dz}{ds} \end{aligned}\right\}$$

$$\therefore \quad \frac{d\frac{dx}{ds}}{\frac{du}{dx}} = \frac{d\frac{dy}{ds}}{\frac{du}{dy}} = \frac{d\frac{dz}{ds}}{\frac{du}{dz}}$$

which proves the proposition.

(X.) A particle moves on a given surface, and is acted on by no forces, but the resistance of the surface show that it will move along the shortest path.

Let $u = 0$ be the equation to the surface,

R = resistance, and $\alpha\,\beta\,\gamma$ the angles which it makes with the axes.

$$\therefore \quad X = R\cos\alpha \qquad Y = R\cos\beta \qquad Z = R\cos\gamma$$

Let

$$V = \left\{\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dx}\right)^2\right\}^{\frac{1}{2}}$$

$$\therefore \quad \operatorname{Cos}\alpha = \frac{\frac{du}{dx}}{V} \qquad \operatorname{Cos}\beta = \frac{\frac{du}{dy}}{V} \qquad \operatorname{Cos}\gamma = \frac{\frac{du}{dz}}{V}$$

$$\left.\begin{aligned} \therefore\ \frac{d^2x}{dt^2} - \mathrm{R}\frac{\frac{du}{dx}}{\mathrm{V}} &= 0 \\ \frac{d^2y}{dt^2} - \mathrm{R}\frac{\frac{du}{dy}}{\mathrm{V}} &= 0 \\ \frac{d^2z}{dt^2} - \mathrm{R}\frac{\frac{du}{dz}}{\mathrm{V}} &= 0 \end{aligned}\right\}$$

Now,

$$\frac{dx}{dt} = \frac{dx}{ds} \cdot \frac{ds}{dt}$$

$$\therefore\ \frac{d^2x}{dt^2} = \frac{d^2x}{ds^2}\frac{ds^2}{dt^2} + \frac{d^2s}{dt^2}\frac{dx}{ds}$$

but since the resistance of the surface is the only acting force, and this is directed along the normal at each point, therefore the force along the tangent plane is equal to zero.

$$\therefore\ \frac{d^2s}{dt^2} = 0$$

$$\left.\begin{aligned} \therefore\ \frac{d^2x}{dt^2} &= \frac{d^2x}{ds^2}\left(\frac{ds}{dt}\right)^2 \\ \frac{d^2y}{dt^2} &= \frac{d^2y}{ds^2}\left(\frac{ds}{dt}\right)^2 \\ \frac{d^2z}{dt^2} &= \frac{d^2z}{ds^2}\left(\frac{ds^2}{dt^2}\right)^2 \end{aligned}\right\}$$

$$\left.\begin{aligned} \therefore\ \frac{d^2x}{ds^2} - \mathrm{R}\frac{\frac{du}{dx}}{\mathrm{V}}\left(\frac{dt}{ds}\right)^2 &= 0 \\ \frac{d^2y}{ds^2} - \mathrm{R}\frac{\frac{du}{dy}}{\mathrm{V}}\left(\frac{dt}{ds}\right)^2 &= 0 \\ \frac{d^2z}{ds^2} - \mathrm{R}\frac{\frac{du}{dz}}{\mathrm{V}}\left(\frac{dt}{ds}\right)^2 &= 0 \end{aligned}\right\}$$

whence

$$\frac{\frac{d^2x}{ds^2}}{\frac{du}{dx}} = \frac{\frac{d^2y}{ds^2}}{\frac{du}{dy}} = \frac{\frac{d^2z}{ds^2}}{\frac{du}{dz}}$$

but $\frac{d^2x}{ds^2}\frac{d^2y}{ds^2}\frac{d^2z}{ds^2}$ are the direction-cosines of the radius of curvature of the curve described by the particle relative to the three axes, which radius lies in the osculating plane, therefore by what has been proved in preceding problems the proposition follows.

(XI.) When a point moves freely in space or on a curve surface, under the action of forces X, Y, Z, such that $Xdx + Ydy + Zdz$ is a complete differential, the path the body takes is such that

$\int vds$ is a minimum,

the integral being taken between the two points at which the motion begins and ends.

$$\left.\begin{aligned}\frac{d^2x}{dt^2} &= X + \frac{R}{M}V\frac{dF}{dx}\\ \frac{d^2y}{dt^2} &= Y + \frac{R}{M}V\frac{dF}{dy}\\ \frac{d^2z}{dt^2} &= Z + \frac{R}{M}V\frac{dF}{dz}\end{aligned}\right\}$$

Let the equation to the surface be

$$F(xyz) = c$$

$$\therefore \frac{dF}{dx}dx + \frac{dF}{dy}dy + \frac{dF}{dz}dz = 0$$

$$V = \frac{1}{\sqrt{\left(\frac{dF}{dx}\right)^2 + \left(\frac{dF}{dy}\right)^2 + \left(\frac{dF}{dz}\right)^2}}$$

$$\therefore 2\frac{dx}{dt}\frac{d^2x}{dt^2} + 2\frac{dy}{dt}\frac{d^2y}{dt^2} + 2\frac{dz}{dt}\frac{d^2z}{dt^2} = 2\{Xdx + Ydy + Zdz\}$$

$$\therefore \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2} = v^2 = 2\Big[\phi(xyz)\Big]_{x'\,y'\,z'}^{x''\,y''\,z''}$$

if $$Xdx + Ydy + Zdz = d\,\phi(xyz)$$

$$\delta\int vds = \int(ds\delta v + v\delta ds)$$

and $$\therefore \delta vds = v\delta vdt = dt\left\{\frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y + \frac{d^2z}{dt^2}\delta z\right\}$$

and $$v\delta ds = \frac{dx}{dt}d\delta x + \frac{dy}{dt}d\delta y + \frac{dz}{dt}d\,\delta z$$

$$\therefore \delta\int vds = \int^{\prime}\left\{\frac{d^2x}{dt}\delta x + \frac{dx}{dt}d\delta x + \frac{d^2y}{dt}\delta y + \frac{dy}{dt}d\delta y + \frac{d^2z}{dt}\delta z + \frac{dz}{dt}d\delta z\right\}$$

$$= \int d\left\{\frac{dx}{dt}\delta x + \frac{dy}{dt}\delta y + \frac{dz}{dt}\delta z\right\}$$

$$= \frac{dx}{dt}\delta x + \frac{dy}{dt}\delta y + \frac{dz}{dt} \text{ between the assigned limit,}$$

$= 0$ because the variations of x, y, z at the limits $= 0$

$\therefore$ $\int vds$ is a minimum.

(XII.) Find the equation to the curve along which, under the action of given forces, a body will move from one point to another in the shortest time.

$$\therefore \quad ds = v\,dt \qquad \therefore \int \frac{ds}{v} \text{ is to be a minimum.}$$

$$\therefore \quad \delta \int \frac{ds}{v} = 0$$

$$\therefore \int \frac{\delta\, ds}{v} - \frac{ds\,\delta v}{v^2} = 0$$

Let X, Y, Z represent the accelerating forces acting on the body.

$$\therefore \quad \frac{d^2x}{dt^2} = \mathrm{X} \qquad \frac{d^2y}{dt^2} = \mathrm{Y} \qquad \frac{d^2z}{dt^2} = \mathrm{Z}$$

$$\therefore \quad vdv = \mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz$$

$$v\delta v = \mathrm{X}\delta x + \mathrm{Y}\delta y + \mathrm{Z}\delta z$$

and
$$\delta ds = \frac{dx}{ds}\,\delta dx + \frac{dy}{ds}\,\delta dy + \frac{dz}{ds}\delta dz$$

$$\therefore \int \left\{ \frac{1}{v}\frac{dx}{ds}\,d\delta x + \frac{1}{v}\frac{dy}{ds}\,d\delta y + \frac{1}{v}\frac{dz}{ds}\,d\delta z \right\} - \frac{ds}{v^3}\left\{ \mathrm{X}\delta x + \mathrm{Y}\delta y + \mathrm{Z}\delta z \right\} = 0$$

$$\therefore \quad \frac{1}{v}\left\{ \frac{dx}{ds}\delta x + \frac{dy}{ds}\delta y + \frac{dz}{ds}\delta z \right\} - \int d\Big(\frac{1}{v}\frac{dx}{ds}\Big)\delta x + d\Big(\frac{1}{v}\frac{dy}{ds}\Big)\delta y + d\Big(\frac{1}{v}\frac{dz}{ds}\Big)\delta z$$

$$- \int \frac{ds}{v^3}(\mathrm{X}\delta x + \mathrm{Y}dy + \mathrm{Z}dz) = 0$$

$$\therefore \quad d\left(\frac{1}{v}\frac{dx}{ds}\right) + \frac{ds}{v^3}\mathrm{X} = 0$$

$$d\left(\frac{1}{v}\frac{dy}{ds}\right) + \frac{ds}{v^3}\mathrm{Y} = 0$$

$$d\left(\frac{1}{v}\frac{dz}{ds}\right) + \frac{ds}{v^3}\mathrm{Z} = 0$$

$$\therefore \quad -\frac{1}{v^2}\frac{dv}{ds}\frac{dx}{ds} + \frac{1}{v}\frac{d^2x}{ds^2} + \frac{\mathrm{X}}{v^3} = 0$$

$$-\frac{1}{v^2}\frac{dv}{ds}\frac{dy}{ds} + \frac{1}{v}\frac{d^2y}{ds^2} + \frac{\mathrm{Y}}{v^3} = 0$$

$$-\frac{1}{v^2}\frac{dv}{ds}\frac{dz}{ds} + \frac{1}{v}\frac{d^2z}{ds^2} + \frac{\mathrm{Z}}{v^3} = 0$$

$$\therefore \quad \frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2 = \left(\frac{1}{\text{Rad. of curvature}}\right)^2$$

$$= \frac{1}{v^2}\frac{dv^2}{ds^2}\left\{ \frac{dx^2}{ds^2} + \frac{dy^2}{ds^2} + \frac{dz^2}{ds^2} \right\} - \frac{2}{v^3}\frac{dv}{ds}\left\{ \mathrm{X}\frac{dx}{ds} + \mathrm{Y}\frac{dy}{ds} + \mathrm{Z}\frac{dz}{ds} \right\} + \frac{\mathrm{X}^2 + \mathrm{Y}^2 + \mathrm{Z}^2}{v^4}$$

$$\therefore \quad \frac{v^4}{\rho^2} = v^2\frac{dv^2}{ds^2} - 2v^2\frac{dv^2}{ds^2} + X^2 + Y^2 + Z^2$$

$$\therefore \quad \frac{v^4}{\rho^2} = X^2 + Y^2 + Z^2 - v^2\frac{dv^2}{ds^2}$$

$$= X^2 + Y^2 + Z^2 - \left\{X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}\right\}^2$$

$$= R^2\left\{1 - \left(\frac{X}{R}\frac{dx}{ds} + \frac{Y}{R}\frac{dy}{ds} + \frac{Z}{R}\frac{dz}{ds}\right)^2\right\}$$

putting $$X^2 + Y^2 + Z^2 = R^2$$

and since $\frac{X}{R}, \frac{Y}{R}, \frac{Z}{R}$ are the direction-cosines of the resultant of the forces, and $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$ of the element of the curve, therefore if φ is equal to the angle between the resultant of the forces and the tangent to the curve,

$$\frac{v^4}{\rho^2} = R^2 \sin^2\phi$$

$$\therefore \quad \frac{v^2}{\rho} = R \sin\phi$$

consequently the pressure on the curve arising from the centrifugal force is equal to that which arises from the impressed forces, then time spent in the description of the trajectory is a minimum. Q. E. F.

(XIII.) Find the curve of a given length, which being suspended from two given points may have its centre of gravity the lowest possible.

Take the axis of x vertical,
consequently we have

$$\int (x + k)\, ds \text{ a minimum}$$

$$\therefore \quad \delta\int(x + k)\, ds = \int\{dxds + (x + k)\,\delta ds\} = 0$$

$$\therefore (x + k)\left\{\frac{dx}{ds}\delta x + \frac{dy}{ds}\delta y\right\} + \int\left[\left\{ds - d\left(\overline{x+k}\frac{dx}{ds}\right)\right\}\delta x - d\left(\overline{x+k}\frac{dy}{ds}\right)\delta y\right] = 0$$

between the given limits,

$$\therefore \quad ds - d\left(\overline{x + k}\frac{dx}{ds}\right) = 0 \qquad\qquad d\left(\overline{x + k}\frac{dy}{ds}\right) = 0$$

$$\therefore \quad \overline{x + k}\frac{dx}{ds} = a + s \qquad\qquad \overline{x - k}\frac{dy}{ds} = b$$

$$\therefore \quad \frac{dx}{dy} = \frac{a+s}{b} \qquad\qquad (x + k)^2 = (a + s)^2 + b^2$$

The equation to the catenary.

If the curve was to hang from some point in a given curve to some point in another given curve, the former part of the resulting expression shows that the curve cuts both the given curves at right angles.

(XIV.) To find the form of a curve which a string of a given length assumes when the enclosed area is a maximum.

$$\delta\int\{ydx + kds\} = 0$$

$$\therefore \int\left\{\delta y\, dx + y\delta\, dx + k\left(\frac{dx}{ds}\delta dx + \frac{dy}{ds}\delta dy\right)\right\} = 0$$

$$\therefore \left(y + k\frac{dx}{ds}\right)\delta x + k\frac{dy}{ds}\delta y + \int\left[\left\{dx - kd\left(\frac{dy}{ds}\right)\right\}\delta y - \left\{dy + kd\frac{dx}{ds}\right\}\delta x\right] = 0$$

$$\therefore \quad k\frac{dy}{ds} = x + a \qquad k\frac{dx}{ds} = -y - b$$

$$\therefore \quad k^2 = (x+a)^2 + (y+b)^2$$

equation to a circle, of which the radius equals k.

(XV.) Find the form of the solid of evolution of given contents which exercises the greatest attraction on a particle in its axis, law varying $\frac{1}{\text{distance}}$.

$$\delta\int 2\pi\left(1 - \frac{x}{\sqrt{x^2+y^2}} + ky^2\right) dx = 0$$

$$2\pi\int\left\{\frac{xy}{(x^2+y^2)^{\frac{3}{2}}} + 2ky\right\}\delta y + \ldots = 0$$

$$\therefore \quad x + 2k\,(x^2+y^2)^{\frac{3}{2}} = 0$$

the equation to the curve which generates the surface of the solid.

Note on Lagrange's Condition for Maxima and Minima of Two Variables.

The conditions that

$$z = f(x, y) \tag{1}$$

have a maximum or minimum value are

$$\left(\frac{dz}{dx}\right) = 0, \qquad \left(\frac{dz}{dy}\right) = 0 \tag{2}$$

and the discriminating condition is

$$\frac{d^2z}{dx^2}\frac{d^2z}{dy^2} - \left(\frac{d^2z}{dxdy}\right)^2 >, =, < 0 \tag{3}$$

in the first of which cases the value determined by (2) will be a minimum, in the last a maximum, and in the second neither one nor the other.

If in (3) we substitute

$$\frac{d^2z}{dx^2} = \xi, \quad \frac{d^2z}{dy^2} = \eta, \quad \frac{d^2z}{dxdy} = \zeta \tag{4}$$

the condition will depend upon the form of the surface

$$\xi\eta - \zeta^2 = \theta \tag{5}$$

where θ is positive in the case of a minimum, negative in that of a maximum, and zero in the case of neither a maximum nor minimum. Now (5) is the equation to a central surface of the second order, having its origin at the centre. To refer this to its principal axes, we have merely to substitute in the usual cubic

$$(P-A)(P-A')(P-A'')-B^2(P-A)-B''(P-A')-B''^2(P-A'')2BB'B''=0 \tag{6}$$

the three roots of which are the coefficients of the squares of the variables in the transformed equation; but since

$$\left.\begin{array}{lll} A = 0, & A' = 0, & A'' = -1 \\ B = 0, & B' = 0, & B'' = 1 \end{array}\right\} \tag{7}$$

(6) becomes

$$(P+1)(P^2-1)=0 \tag{8}$$

the roots of which are -1, -1, and 1, hence the transformed equation becomes

$$x^2+y^2-z^2=-\theta \tag{9}$$

which is consequently in the case of a

minimum, a hyperboloid of revolution of two sheets,
maximum, ,, ,, one sheet,
neutral result, a cone.

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